THE CARUS MATHEMATICAL MONOGRAPHS.

A REPORT OF PROGRESS BY H. E. SLAUGHT.

In the Autumn of 1921, Mrs. Mary Hegeler Carus made a notable gift to the Mathematical Association of America in the form of an annual contribution of $1,200 for five years, beginning in January, 1922, for the purpose of enabling the Association to prepare and publish a series of mathematical monographs. The full text of the deed of gift and acceptance by the Trustees of the Association is printed in this MONTHLY for October, 1921, pages 353, 360.

The purpose in the mind of Mrs. Carus, as indicated both by word of mouth and in her written communications, is to popularize mathematical intelligence by making accessible at nominal cost a series of expository presentations of the best thoughts and keenest researches in the field of mathematics. These presentations are to be set forth in booklet form in a manner comprehensible not only to teachers and students specializing in mathematics, but to scientific workers in other lines. It is furthermore desired to reach that still wider circle of thoughtful people who, having a moderate acquaintance with elementary mathematics, are quite willing and eager to extend that acquaintance indefinitely along informational lines, provided it can be done without prolonged and painful study of the mathematical treatises which abound in extreme rigor and endless detail.

The Trustees immediately appointed a committee consisting of Professors W. D. Cairns, Oswald Veblen and H. E. Slaught, "to nominate to the Trustees an editorial board on monographs and to formulate a statement of powers of this board." This committee, having no instructions other than those just quoted, found itself confronted with a very serious responsibility, for it recognized that upon the wisdom of its action depended in large measure the success or failure of this enterprise. Many questions at once arose: Should the editorial board contain a large number or a small number of members? Should they be widely separated geographically or concentrated near one center? Should the members represent all the leading divisions of mathematics,—analysis, geometry, number theory, etc.? Should this board be empowered to act in an editorial capacity only or should its members be chosen as well for their fitness to prepare some of the monographs themselves? These and numerous other questions required most careful consideration and led to much consultation and extended correspondence.

After prolonged study of these questions, the nominating committee reached the following conclusions:

(1) The editorial board should be limited initially to three members and should be so chosen that they can actually get together for personal conference with little expense and loss of time. This need has been demonstrated in the case of other committees, but seems to be imperative in this case since an entirely new project has to be worked out, one which could hardly be done by correspondence or widely separated meetings.

(2) This central body should have full and final authority in selecting authors, passing upon manuscripts and all other matters pertaining to the preparation and publication of these monographs, subject only to restraint or veto of the Board of Trustees. They should, however, be authorized to call into service any members of the Association, especially in viséing manuscripts, reading proof, or supplying any advice or assistance that may be needed, assuming that this is a work of public welfare in mathematics and that any member of the Association should consider it his patriotic duty to serve in whatever capacity this editorial board may call upon him.

(3) One member of this editorial board should be chosen for his qualifications as a business manager, since many questions of a business character are bound to arise at every turn. The other two should be chosen not only for their editorial qualifications, but also for their ability and fitness actually to produce the first two monographs and this they should be instructed to do.

The reasons for this latter recommendation are self-evident upon reflection. A somewhat careful examination of the numerous existing series of mathematical monographs in Europe and the few in this country shows that none of them are designed to fulfill completely, and most of them not even approximately, the purpose indicated for the Carus Monographs, and hence this editorial board has a new and exceedingly difficult task to perform. For this reason the two author-editors should be particularly well qualified to collaborate in working out the desired form of presentation and in setting the standard for future numbers of the series.

(4) It goes without saying that these two men must have unquestioned scientific standing in the mathematical world and particularly throughout this country and that they must have had successful editorial experience in the mathematical field, together with a good appreciation of literary qualities, especially as to clear and forceful use of English. Moreover, these men must have firmness and independence sufficient to reject any manuscript from whatever source if it does not conform to the standards set up for these monographs; they must have the tact and good sense which will enable them to deal with authors in such a way as to bring out in these publications the very best that is possible through suggestions and criticism. They must have a keen appreciation of the aim and scope of these proposed monographs as intended by Mrs. Carus and an enthusiastic determination to see these objectives realized to the fullest possible extent. And finally they should be willing to enter upon a five-year program, since it is hardly possible that the success of this undertaking can be demonstrated in a shorter time, and surely one of the most important factors is continuity of purpose and control.

(5) Taking into full consideration all of the foregoing specifications, the nominating committee considered itself exceedingly fortunate in being able to secure the acceptance of Professors David Raymond Curtiss of Northwestern University and Gilbert Ames Bliss of the University of Chicago as the author-editor members of the editorial board for the Carus Monographs. The committee also insisted by a two-thirds vote that Professor H. E. Slaught be selected as the business manager of the editorial board.

The foregoing recommendations were presented in oral outline to the Board of Trustees of the Association and approved at the annual meeting in Cambridge in December, 1922. The present elaboration of these recommendations may be considered a part of the nominating committee's report and a copy of the same will be filed with the minutes of the Cambridge meeting.

The Editorial Board has held numerous meetings and has agreed upon certain general policies and procedure, as well as upon many details, some of which are herewith given.

(1) The choice of subjects for the early monographs was a question much in the minds of all who gave the matter any serious thought. This choice was at once limited, as was intended, to topics in analysis by the selection of Professors Curtiss and Bliss as the author-editors of monographs one and two. After careful consideration, it was agreed that Professor Curtiss would write on "Functions of a Complex Variable" and Professor Bliss on "Calculus of Variations," and that they would collaborate in the closest possible manner on all questions of scope and form of presentation.

(2) It is agreed that for the present, and until pretty definite standards have been established, no general call will be made for authors to present completed manuscripts for consideration and acceptance, which procedure might easily swamp the editors at a time when all their energy is needed in the actual determination and working out of these standards. On the other hand, the editors are carefully considering the selection of other authors with whom they believe effective collaboration can be established most surely and directly, and whose selection will insure a proper variety and range of topics.

(3) The editors do wish most emphatically to appeal for general and widespread coöperation on the following lines:

(*a*) They desire lists of topics suggested for monographs, arranged in the order of precedence preferred.

(*b*) They desire names of persons who are believed to be especially competent to prepare such monographs together with reasons in detail for such belief.

(*c*) Any person interested in the preparation of a monograph is earnestly requested to communicate with the editorial board, stating the topic proposed to be treated, giving an outline of the proposed treatment, and giving a sample chapter all in typewritten form. Such procedure will enable prospective authors and the editors to become acquainted on a practical working basis with the least possible delay and with no chance for misunderstanding or embarrassment.

It is agreed to be desirable in general that a monograph should be limited to about 125 to 150 pages, size $3\frac{1}{2}$ by $5\frac{3}{4}$ inches, printed with nine point leaded type, that there should be a minimum of display formulas and equations, and that the expository form of presentation should be used as much as possible. The books are intended primarily for the private reader rather than for the class room.

(4) For the benefit of prospective authors, the editors wish to suggest that it will be desirable in the preface of each monograph to state explicitly the maxi-

mum mathematical attainment on the part of the reader pre-supposed by the author and that the author should adhere religiously to this agreement at every point. For example, the editors have agreed that in the first two monographs this maximum requirement shall be an introductory course in the Calculus. In the case of other subjects, the maximum may doubtless be made lower and should be whenever feasible; while in all cases it will be the purpose to keep the prerequisites within the range of the widest possible circle of readers, certainly including all teachers of mathematics in high schools and colleges and all students and general readers who are acquainted with the usual undergraduate courses in mathematics.

Naturally, such monographs cannot contain rigorous proofs of all theorems presented, but they can give a readable exposition of the significance of such theorems with definite references where rigorous proofs may be found, and they can show how these theorems are used to build up in logical order the underlying doctrine of each subject, how this doctrine dovetails with that of other subjects, and how the whole fabric of mathematical truth, as related to this subject, finds useful and powerful applications in scientific affairs. Such monographs intended primarily for very moderately informed readers can be produced successfully only by the highest skill of the best informed writers.

(5) It is agreed, with the consent of Mrs. Carus, that a certain part of the fund shall be used to remunerate authors for clerical service, typewriting, etc., needed in connection with the preparation of monographs. It is further agreed that, in lieu of royalty payments to authors from the sale of the monographs, a certain lump sum shall be paid as an honorarium to an author out of the sales income from the first thousand copies and that this amount may be augmented by additional payments to be determined by the rate of sale realized beyond the first thousand.

It is confidently believed that at least one thousand copies of each monograph will be sold to the members of the Association to whom the price will be adjusted on the cost basis. The distribution to the general public will be through business channels such as the Open Court Publishing Company of Chicago, in whose general catalog of publications, which has a very wide circulation both in this country and in Europe, announcement has already been made of this projected series under the auspices of the Mathematical Association of America.

Finally, it should be said that the range of such a series of monographs is as wide as the field of mathematics itself. There may be fifty, or one hundred, or even two hundred topics in pure and applied mathematics, including historical and biographical subjects, which are capable of successful treatment in this series. At any rate, since the generous donor proposes to endow this fund, if the plan proves to be feasible and practicable after five years of preliminary trial, the only cause for eventual cessation will be either lack of appropriate topics or scarcity of qualified authors.

There is no intention of limiting the work of preparing these monographs to American authors. The editors will look with great pleasure for suggestions from

or concerning authors in any part of the world who may be interested in this project, the only condition being that an accepted manuscript shall be prepared and printed in English.

ON THE SUBJECT MATTER OF A COURSE IN MATHEMATICAL STATISTICS.[1]

By H. L. RIETZ, University of Iowa.

1. Introduction. It was suggested by Professor Huntington as Chairman of our Program Committee that this paper give, among other things, a brief report on such a partial survey of existing courses in mathematical statistics as I could make in the short time available to obtain the necessary information. My first step in attempting such a partial survey consisted in an examination of the recent catalogue announcements of a large number of colleges and universities. These announcements show that many departments of mathematics offer at least a beginning course in statistics, and that a relatively small number of departments—seven among the institutions whose catalogues I examined—offer also an advanced course in mathematical statistics. My next step consisted in writing letters of inquiry to a number of teachers of the courses in statistics. This correspondence disclosed the fact that the courses differ so widely in their prerequisites, organization, and content as to make it difficult to present a precise report without going into details. But neglecting details, we find at one extreme certain experiments in the teaching of freshmen courses in statistics. At the other extreme, we find advanced courses in mathematical statistics taken primarily by graduate students of mathematics. Although I have been unable, in general, to form more than a rough judgment as to the content of the courses as given in a particular department, it appears that the courses differ so much as to be widely scattered over the interval between the two extreme types of courses just mentioned. Moreover, it is but natural to expect that these relatively new courses based on material selected from a large and rapidly growing field would show much variation both as to prerequisites and subject matter.

As the subject matter of a course in mathematical statistics is determined to a considerable extent by the influences which have led to the development of such courses and by the primary aims of the course, let me attempt to describe briefly these influences and what I conceive to be the chief aims of such a course. The influences which have led to the development of the courses have come from at least two sources. A wholesome influence has come from the revival of interest in the pure theory of probability. This revival of interest is expressed both in recent papers on probability theory and by the appearance of books on probability by Poincaré, Borel, Bachelier, Czuber, Fisher, and Keynes. The main influence, however, in the development of courses in mathematical statistics has come from the demand for better statistical methods in handling the data of the rapidly

[1] Presented before the Association at the Symposium held in Cambridge December 29, 1922.

growing statistical sciences. The dominant factor is thus to be found on the side of the applications.

To illustrate the operation of the influence of practical statistics on the development of mathematical statistics, consider the situation of Francis Galton when he was working on the problem of likeness of parents and offspring with respect to human stature. His intuition for statistical inference was well developed. It was no doubt fairly obvious to him from a rough survey of his data on family history that there is neither the kind of perfect dependence between the heights of father and son that is given by a simple usable mathematical function nor the independence ordinarily assumed in probability theory. Galton thought that the analysis of his data presented an important mathematical problem. His suggestion of the problem to J. Hamilton Dickson, Karl Pearson and others initiated a development of correlation theory which is now being applied to a remarkably wide range of data.

The influences which have led to the development of courses in mathematical statistics are closely connected with the chief aims in the teaching of the subject; and I shall next attempt to make a statement of what seem to me to be the leading aims in the teaching of mathematical statistics. It seems to me that one aim is to develop in the minds of the students the meanings of what I shall call appropriate norms or patterns for the elucidation of the mass phenomena to which statistical data apply. The development of such patterns in the mind seems to me to be as fundamentally necessary for the description of statistical data as is the development of the appropriate elementary geometric forms or patterns in the mind of the student for the description of the shapes and sizes of objects in the space of experience. In statistical analysis, the patterns are represented by binomial distributions, probability curves, Bernoulli, Poisson and Lexis series, tetrachroic functions, regression curves, correlation surfaces, and so on.

The second important aim in the teaching of mathematical statistics is to develop the principles and theorems for the solution of practical problems involving the determination of significant differences and relations among statistical variables. The solution of these real problems consists largely in giving a satisfactory description of a collection of items. In his recent book, Keynes says that the first function of the theory of statistics is purely descriptive. While this statement seems to be correct, it is also correct to say that appropriate mathematico-statistical methods go much further than an ordinary non-mathematical description. The mathematical analysis makes possible inferences which could in all probability not have been otherwise made and it should offer safeguards against incorrect inferences and pitfalls. In this use of mathematical statistics there are, however, some dangers. The greatest danger seems to me to be the failure of some persons applying formulas to recognize the limitations of the formulas they are using. Thus, the formality of the method of measuring correlation by a coefficient is likely to be carried too far. For instance, let me cite the particular case where a few years ago I gave some lectures on correlation

before some instructors in agriculture. Presently I found one of them so taken up with correlating things in his farm survey work that I told the dean of the college that we must stop this man "correlating."

With the above mentioned aims in mind, suppose we attempt to determine the subject matter of a course. We do not get far until we are confronted with the problem of selecting the subject matter adapted to the mathematical preparations of the rather heterogeneous group of students from whom the demand for the course comes. Thus, we have to consider the subject of prerequisites. While the demand for the course comes from many classes of students with respect to mathematical preparation, it has seemed to me to be good educational policy and fair economy to recognize especially two classes of students with respect to qualifications to enter upon the study of statistics. The one group consists of those who have had training in college mathematics at least through a thorough first course in calculus and have the degree of maturity of superior seniors or graduate students. The other group consists of the rather mature students, say juniors, seniors, and graduate students from departments such as economics, biology, and psychology. These students are often dealing with statistical data but are lacking entirely or almost entirely in training in college mathematics. The two classes of students thus recognized would include no freshmen or sophomores. It is not unlikely that there are situations in which it is also desirable to give statistics in the freshman or sophomore year, but ordinarily I fear such a policy is likely to result in side-tracking a good many students from the regular mathematical courses which they should be encouraged to take.

Since it is obvious that a much better course can be given to students with good mathematical training than to those without such training, the very natural reaction of the mathematics teacher would be to dispose of the problem by specifying a thorough first course in calculus as a prerequisite for any course in mathematical statistics. Let me say that I wish this were the practical solution of the whole problem, and perhaps it will in time be the solution, but we are confronted with a condition which calls for consideration. Under actual conditions, this view of requiring calculus as a minimum prerequisite for any course in statistics is not so practicable nor perhaps so generous in meeting a real situation as it may seem to be at first reaction. The actual situation which has confronted me and perhaps most teachers of mathematical statistics is a demand from seniors and graduate students of economics, biology, psychology and other departments interested in some statistical project for which they feel the need of better statistical methods. It is hardly necessary to say that such a group of students is usually represented by suitable attorneys in the form of professors in these various departments, who urge that their students should have the course.

When the interested student has had at least a year of college mathematics, it has been my experience that he can often be convinced that it is to his best interest to take a course in calculus before entering a course in statistics, but when he has had no college mathematics, he usually feels that it is impracticable

for him to take all the preliminary mathematics up to and through a course in calculus. Moreover, it must be granted that a substantial amount can be taught about statistical methods without requiring calculus as a prerequisite. To meet the situation, certain departments are giving two kinds of courses in statistics. They are giving a sort of self-subsisting (*selbständig*) course for students who are interested in the analysis of data, but have had little or no college mathematics, and they are giving another course requiring a substantial amount of mathematics as a prerequisite. In what follows, I shall for convenience call the former an elementary course and the latter an advanced course.

To illustrate from my experience, let me say that I have for some time been giving two courses of three hours each throughout the academic year. The elementary course requires no college mathematics as a prerequisite, but at least junior standing in college. It has been taken by superior juniors and seniors and graduate students from departments other than mathematics. The course may be described as a combination of statistics and of necessary mathematical preliminaries adapted to rather mature students interested in the analysis of data. The elementary and advanced courses alternate from year to year. In the present year, I am giving the advanced course to eleven students of whom ten are graduates and one is a senior. Except for one graduate student of education who has had very little mathematics beyond a first course in calculus, each student has had several courses in mathematics beyond the sophomore calculus.

Last year slightly over half the students of the elementary course were juniors and seniors of the department of economics, and the remainder of the group was made up of graduate students from economics, biology, and psychology. I mention these classes of students in order to give a sort of picture of the kind of students interested in these courses.

The elementary course to which I have referred is conducted by having two lectures or class periods a week, and three hours in the statistics laboratory provided with computing machines. Such a combination of class room and laboratory work I regard as a distinct success in handing applications to real statistical problems.

Although it is not my intention to mark out a course to the extent of giving a syllabus, it has seemed to me that our discussion would probably be stimulated by giving a pretty definite notion of the subject matter of the two types of courses which I carry in my mind, with special reference to places for emphasis in the teaching of certain topics, and to certain points which may provoke discussion; for I understand it is my duty to be provocative on this occasion even at some risk of not being able to defend my position in all cases. It seems convenient to group my remarks and comments around certain groups of topics somewhat like chapter headings. Perhaps I should say that these groups of topics are not selected for purposes of logical organization of material, nor in all cases to indicate an order of presentation, but largely for convenience in shortening my comments —a subject in which I assume we are all interested.

2. Elementary course. Apart from a brief historical introduction, and a

few lectures on the chief sources of statistics, the course begins with an elementary treatment of frequency distributions, averages and measures of dispersion. Even at the risk of being tedious, let me explain to a slight extent the procedure in starting this course.

The work is begun by giving each student a sheet of data, say the monthly rainfalls of Iowa City for the past twenty-five years, and asking him to write out any facts of interest which he can draw from the data by ordinary common sense observation. After defining frequency distribution our next step is to require each student to exhibit the rainfall data in the form of a frequency distribution, and to make another frequency distribution by throwing coins or dice, say you have them carry out the experiment of throwing sets of seven coins 128 times making a frequency distribution of the number of heads per throw. Carry these concrete problems both from actual statistics and from games of chance through the process of constructing frequency polygons and freehand frequency curves.

The elementary treatment of different kinds of averages seems to me to be pretty well standardized as is shown by an examination of a number of textbooks. It may, however, be worth while to emphasize the teaching of short methods of computation and appropriate forms for computation. With these students who are inexperienced in numerical computation, special consideration should be given to such questions as the number of significant figures to be retained, to the relative errors in products and quotients, and to other questions of accuracy in numerical calculations. Another place for emphasis is in the selection of suitable concrete problems to illustrate the adaptation of an average to a particular purpose.

Since much of the discussion of the use of different types of averages has taken place in the preparation of index numbers, it seems desirable to give some simple applications to index numbers although the methods of preparation of index numbers should constitute a later chapter where we should lead up to the "best average" for index numbers as given recently by Irving Fisher, and as supported in a careful analysis by Allyn A. Young.

Let us consider next diagrammatic and graphical representation. I have sometimes thought that too much time is likely to be spent in making attractive pictures about obvious comparisons and relations. The Committee on Mathematical Requirements has very properly recommended that some graphical representation of statistics be taught in high schools. However, by a careful selection of material, I feel sure that my students have found a brief treatment of this topic of value. For practice exercises, I have found it useful to go through W. C. Brinton's *Graphical Methods of Presenting Facts* (New York, 1914), and make class exercises to illustrate practically every important point on graphical representation discussed in this book. It requires but short time to do these exercises and I think the students have found this work profitable. Among the large variety of detailed schemes, there is, of course, included the plotting of statistical variables on coördinate paper. A statistical curve with time as the abscissas supplies a very simple concrete basis for interpolation. This graphical

work leads naturally into the graphical representation of a mathematical function. Indeed the next chapter deals with the graphical representation of mathematical functions and the subsequent two groups of topics may be fairly well described as pure mathematical preliminaries to further progress in the course.

It is just as desirable in mathematical statistics as in other fields of mathematics that the meaning of mathematical function be clear, and that the pictures of changes in the function which correspond to changes in the assigned values of the variable be vivid. The tables of corresponding values in the case of the mathematical function are so analogous to certain statistical tables that we have a good setting for the development of the notion of a mathematical function. This graphical representation early in the course should be carried at least as far as it is carried in those of our college algebras which emphasize the subject, and I should carry the work somewhat further by developing the equation of a straight line to satisfy given conditions, and by finding the parabola $y = a + bx + cx^2$ through three points.

Let us consider next interpolation and graduation. Interpolation is so much used in statistical analysis that its treatment should be carried as far as the method can be appreciated by these students with no training in college mathematics. With the equation of the straight line through two points given it seems well to make graphical ideas the basis of interpolation by proportional parts. In my experience, it has been found practicable to introduce higher differences in this course, and to teach something of the meaning and use of both Newton's and Lagrange's interpolation formulas. Some of the simpler methods of graduation of certain data so as to remove minor irregularities may be taught very naturally immediately following interpolation.

Let us turn next to permutations, combinations, the binomial theorem, the elements of probability, logarithms, logarithmic and exponential functions. These topics hardly require any comment, except possibly the very general one to the effect that the more thoroughly the elements of these topics are studied early in the course, the more is likely to be accomplished later with practical problems of statistics. One special comment may be worth while to say that emphasis should be placed on statistical probability, on experiments with urn schemata, and on the uses of logarithmic and semi-logarithmic paper.

The study of the elements of probability leads us naturally to the binomial distribution. It seems practicable to me to follow Yule's chapter on this topic pretty closely in demonstrating that the standard deviation of the binomial distribution $(p + q)^s$ of the frequencies is $\sqrt{spq}$ and of the corresponding relative frequencies is $\sqrt{pq/s}$. It is important that the students experiment in making distributions by the use of urn schemata, and compare their results with the *a priori* most probable values.

The binomial distribution forms a natural introduction to the normal probability curve. It is surely of first rate importance that the student appreciate not only the value of the normal curve in statistics, but also the limitations on its applications in the description of frequency distributions. For the purposes

of this class, the curve may well be obtained as a limiting case of a symmetrical binomial distribution. It should be held prominently before the students that the normal probability function thus obtained represents a very special law of probability, and sufficient practice problems should be given in fitting the curve to actual distributions to make sure that the student makes proper application of both the tables of the probability integral and of the ordinates of the curve in calculating the theoretical frequencies given by the normal curve. The failure to fit reasonably well most of the given distributions forms the basis of a natural introduction to generalized frequency curves, but it seems impracticable to attempt more than a graphic treatment of generalized frequency curves in this elementary course.

Following the study of the normal probability curve, we may well give a sort of informal introduction to sampling theory. For example, it seems appropriate to introduce at this point in the course a suitable selection of material from such a chapter as that of D. C. Jones on probability and sampling which deals with the meaning of probable error, and with the derivation of the probable error in a simple relative frequency and in an arithmetic mean.

Let us consider next the Lexis theory of dispersion of relative frequencies. My class last year responded with enthusiasm to the classification of series according to the Lexis theory. Each member of the class carried out several experiments on urn schemata to give a concrete picture of the Bernoulli, Poisson, and Lexis types of dispersion. This theory of Lexis may be described as a method of breaking up statistical distributions into selected sets of sub-series, with a view to analyzing the stability of the relative frequencies among the sub-series. It is only the part of common sense to assume that the breaking of a statistical series into sub-series for examination would facilitate analysis, and form an important safeguard against erroneous conclusions from averages based on the aggregate only. The three types of urn schemata in the Lexis theory constitute a useful set of norms for setting up analogies with actual statistical series.

Let us turn next to the correlation theory. The treatment of correlation theory should be made one of the main topics of the course. The preparation of convenient forms for numerical calculation of the correlation coefficient should be stressed; for the inaccuracy of the beginner in the calculation of correlation coefficients is usually appalling. It is of first rate importance to emphasize the meaning and limitations of the correlation coefficient both from a common sense and from a precise technical standpoint. Bring out the fact by illustrations that the use of a summary formula like the correlation coefficient does not relieve us entirely of the necessity of close investigation of the data in subgroups. Make much of the mean-square-error of estimate—that is, of the standard deviation of arrays of the correlation table. Misunderstandings and extravagant interpretations of the accuracy of prediction by the use of a correlation coefficient arise from lack of appreciation of this point. For example, I feel sure that certain college administrators, who do not know correlation theory, fail to realize with what little security you can predict the achievement of an individual student

from an intelligence test because there is found to be a correlation of .6 between the results of such a test and actual achievement in college work. As you well know, the facts are that with $r = .6$ the average standard deviation of an array under the most ideal conditions is .8 of the standard deviation of the whole group, and it would obviously be a grave injustice to an individual to judge his achievement from such a test. On the other hand, this correlation of .6 based on large numbers gives, under the same ideal condition of linear regression, a close prediction as to the average achievement of those who make an assigned grade in the test.

The chief way that has occurred to me to guard against the dangers of extravagant interpretations of correlation coefficients is to give a diversity of applications, and have the student carry several problems through to completion and formulate conclusions, subject to limitations on the particular problem in hand. The correlation ratio should be as fully and carefully treated as the correlation coefficient, and the form for its calculation may well accompany that for finding the correlation coefficient as is shown in a paper by Crathorne in the *Journal of the American Statistical Association* for September, 1922.

To show what correlation means in relation to urn schemata, I have found it a valuable exercise for these students to experiment with urn schemata with certain limitations on independence in the probability sense, and investigate the correlation from data thus obtained.

With the group of students from psychology, it seems important to teach Pearson's treatment of correlation in ranks, and to point out the meaning of Spearman's "footrule"; for students of psychology are almost sure to ask whether Spearman's footrule is not good enough for measuring correlation when only small numbers are available.

At least the meaning of partial and multiple correlation should be presented, and I should carry the class as far in this topic as their mathematical preparation permits. You are perhaps aware that educational psychologists are at present making considerable use of partial correlation methods, and in this connection it is surely important to emphasize the meaning and limitations of the methods. It is practicable and desirable to carry through some simple applications of the correlation of deviations from a line of general trend, thus giving an introduction to correlation in time series.

As perhaps the final topic in the course, it is important to return to a consideration of random sampling. The idea of random sampling has been developed to a slight extent in an earlier chapter, but it seems that more should be done with it. The students in this course are almost sure to bring up questions which make it desirable to discuss the meaning of Pearson's criterion of goodness of fit and to apply it to some of the problems we have had earlier in the course where we passed judgment on goodness of fit mainly by a common-sense process. It needs hardly be said here that the limitations of the method should be especially stressed with this class of students.

In bringing to a close my comments on the elementary course, let me direct special attention to the fact that in this course theory is taught in so far as it is

practicable to do so in carrying out the aims of the course, but much is made of concrete problems and experiments in statistical probability. In closing my remarks on this course let me say that by the elimination of the chapters on mathematical preliminaries, the course seems well adapted to mature students who have had freshmen mathematics. Such students may well take the course with a reduction in credit on account of such duplication of work as is likely to occur.

3. Advanced course. Turning now to a brief consideration of the advanced course, there is hardly a subject in the elementary course which cannot be greatly enriched when the students taking the course have adequate mathematical preparation to be introduced gracefully to the theory of statistics. In my comments on this course, I shall follow to some extent the order of topics of the elementary course, and interpolate additional topics.

Our treatment of averages in the elementary course is one to which we might conceivably care to make but slight extension, but even here the introduction of the average of an infinite number of elements, the identity of centroid and arithmetic mean, of the radius of gyration and standard deviation, set up associations of real interest to the student.

It was suggested that the treatment of binomial distributions in the elementary course might follow Yule rather closely. In the advanced course, it is fairly clear that a treatment similar to that in E. Czuber's *Wahrscheinlichkeitsrechnung* (Leipzig, 1910–1914) or to that in A. Fisher's *Mathematical Theory of Probabilities* (New York, 1922) is to be preferred. Closely associated with the binomial distribution is the theorem of Bernoulli. My experience indicates that this central theorem may well be developed in practically the form in which it is stated in the German and French encyclopedias of mathematics, although this statement includes much in addition to the original statement of Bernoulli. With respect to the consideration of the well known controversy about the application of the principle of Bayes[1] in the derivation of the converse theorem, it seems best to me to do little more than direct attention to the controversy at this point, and to defer its careful consideration until near the end of the course.

A substantial chapter on interpolation by the use of higher differences should be among the early subjects in this course and this chapter may well be followed by one on the methods of graduation of statistical data. The principles of the methods of least squares and of moments should likewise be taught early in the course with some simple applications to curve fitting. Certain quadrature formulas should be developed for the application of the method of moments of areas. In particular, Sheppard's corrections of raw moments of frequencies should be derived.

The treatment of the normal probability curve in the elementary course is an almost purely experimental treatment. In the advanced course, such experiments on fitting the curve to data are important, but the various derivations of

[1] Compare this MONTHLY, *1920*, 347–354: "The average reading vocabulary; an application of Bayes's theorem."—EDITORS.

this law of probability, such as those of Gauss, Hagen, Morgan-Crofton, Thompson and Tait, and others, may be profitably studied. Particular emphasis should be placed on the assumptions on which each derivation rests, and on the question of our knowledge or lack of knowledge as to whether the assumptions are valid for the real problems of statistics. After this study of the normal probability function, it seems appropriate to take up the Poisson-exponential—the Bortkewitsch "law of small numbers." This law applies to cases in which the probability of occurrence is very small. Applications should be given to fit the Poisson-exponential to distributions such as the number of deaths per month or year from a minor cause, or to distributions such as that of the Rutherford and Geiger experiment (*Philosophical Magazine*, series 6, volume 20, 1910, p. 698) on the frequency of the number of α-particles radiated from a disc per one eighth second.

It is interesting to experiment with this law in comparison with the Gaussian law of probability in assigning deviations above and below a certain most probable value. The criticisms of Miss Lucy Whitaker of the Bortkewitsch representation should receive attention, although it seems to me her main criticism is invalid.

In this advanced course, there should be included a thorough treatment of generalized frequency curves. With both the Pearson system of generalized frequency curves and what I shall call the Charlier system available, it is a question of difficulty for some of us to decide upon the most appropriate treatment of the subject. It seems to me that the derivations of both systems should be studied in this course, with emphasis first on the character of the assumptions from which they are developed. Then the student should make a variety of applications of both systems to actual distributions as experiments in the facility with which each system applies. My students have shown a good deal of interest in using both systems, but I am not ready to announce a general conclusion from these experiments. I will say that there remains no doubt in my mind that the Charlier system is born in a higher region of mathematical thought than the Pearson system, but the final test consists in trying these systems on many classes of data to see how well they work. The Charlier system has not had extensive use in this country largely because the auxiliary tables were not easily accessible. Fortunately, Fisher in his 1922 edition of *Mathematical Theory of Probabilities* has given such tables to four places, and more extensive tables are given by Glover in his *Tables of Applied Mathematics*, 1923. Among frequency curves deserving special consideration are also those arising by the transformation of the independent variable of well-known frequency curves.

The Lexis theory of dispersion should be studied thoroughly in this advanced course. This study should include the generalization by Coolidge in the *Bulletin of the American Mathematical Society*, volume 27, pp. 439–442, 1921, in which he generalizes the averages which fall under the Lexis theory, and makes the probabilities defined by Lexis a special case.

In correlation theory, we have one of our chief opportunities to extend the work in the elementary course. This should involve the development of the

theories of non-linear regression, and of partial and multiple correlation. It should include the methods of Persons and of Crum for the study of the correlation of time series. It is also of interest to study the correlation surfaces in three and higher dimensions and the curves and surfaces of equal distribution. Another set of configurations of interest which may well be included are the curves and surfaces with the density of distribution which exists under independence.

In the treatment of random sampling we are able to do much in this advanced course where we could do very little but discuss meanings in the other course. After developing the theory of probable errors of various frequency constants, it seems especially important to study the work of Student, Soper and Fisher in *Biometrika*, volumes 6, 9 and 10, on the distribution of means, standard deviations, and correlation coefficients from small samples. The development of the Tchebycheff's inequality should be included because the distribution in this case can be a more or less arbitrary continuous function. As a final section on sampling theory, Pearson's criterion of goodness of fit of theory and observation (*Philosophical Magazine*, 1900) should be developed.

The applications of harmonic analysis to statistical data including the periodogram method should be treated in this course by carrying through the numerical work on some actual data.

Near the end of the course, it is well to deal with the present status of such controversial questions as the validity of the principle of Bayes on inverse probabilities. This question is again of special interest at present because Professor E. T. Whittaker gave in the *Transactions of the Faculty of Actuaries*, volume VIII, 1920, p. 163, a very clear argument to show the invalidity of Chrystal's well-known criticism of the inverse theory, and Karl Pearson gave also in *Biometrika*, volume XII, 1920, p. 11, the derivation of a sort of generalized result on inverse probability which does not make the much criticized assumption of equal distribution of the unknown probability.

If the time were available for the introduction of additional topics, it seems appropriate to include the mathematical theory of the flow of populations, and the mathematical theory of risk, including Landre's theory of maximum risk. Possibly a chapter earlier in the course should be given to the analysis of mortality statistics, but it has been my assumption that this important topic may be appropriately regarded as a special one to be included in a course in actuarial theory, or in a special course in vital statistics. It is desirable, however, to draw upon population and mortality statistics for applications. The analysis of mortality statistics naturally suggests the relation of courses in actuarial theory to courses in the general theory of statistics. Actuarial theory is, of course, concerned with a special field. Although I am much interested in the teaching of actuarial theory, it seems doubtful whether it would be economical for more than a small number of institutions to offer courses in this special field. On the other hand, it seems to me that a general course in mathematical statistics touches such a diversity of interests and applications that it would be justifiable for all large institutions to develop courses in the subject.

In conclusion, let me say that the courses which I have tried to describe are changing from year to year with new developments and are in the very nature of the case defective in many respects. It is my hope that the discussion of this paper will supply some of these defects, and point the way to progress. As to the outlook for further progress in the movement to develop courses in statistics, let me say first that in the eighteen years in which I have given such courses, there has been a healthy growth in interest and also a great improvement in the subject matter of the courses, and I see no indications that the movement will come at once to a standstill. In fact, I look forward to great improvement in these courses in the future, because they are concerned with the development of those ideal patterns which form the very basis of precise statistical description, and because there is every reason to expect an increasing demand for better statistical analysis along with the ever-increasing tendency to enter upon scientific projects which involve the accumulation of statistical data.

HISTORICAL-MATHEMATICAL PARIS.[1]

By DAVID EUGENE SMITH, Columbia University.

III. La Rive Droite.

Few of the older mathematicians had abiding places on the right bank of the Seine, most of them, as already seen, preferring the Quartier Latin. Guillaume Budé,[2] the prime mover in the founding of the Collège de France, however, died at No. 203 of Rue Saint-Martin, as an inscription shows. This is a long street running parallel to the Boulevard de Strasbourg from near the Conservatoire des Arts et Métiers to the Seine. Budé was one of the leading scholars of his day and wrote a work *De asse et partibus ejus libri V* (1516) which was very well known.

Even the Louvre, which we ordinarily look upon as the ancient abode of royalty and the modern home of the graphic and plastic arts, has a certain connection with mathematics. In the years immediately preceding the Revolution it was the seat of the academies and the lodging place of certain savants. Even under the first empire it served in the latter capacity, as witness a letter from Legendre in which he begs for quarters for himself in the older part of the palace. Lagrange[3] lived at No. 124, Rue du Faubourg-Saint-Honoré, a street which, after crossing the Rue Royale, bears the shorter name of Rue Saint-Honoré. A little to the north, near the Gare Saint-Lazare, the Hungarian pianist, Franz Liszt, lived at No. 63, Rue de Provence, and took an interest in the arithmetical prodigy, Henri Mondeux.[4] Among my autographs is an invitation to a musical séance which Liszt gave "au bénéfice du jeune Pâtre mathématicien" on April 30, 1841, with a page of Mondeux's writing on the back.

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[1] This is the conclusion of an article which begins on page 107 of the March–April number.

[2] Born at Paris in 1467; died there in 1540.

[3] Joseph-Louis, Comte Lagrange, born at Turin in 1736; died at Paris in 1813. He was probably the greatest all-round mathematician of his time.

[4] He died in 1861.

In the vicinity of the ancient Temple, at No. 41 of Rue de Bretagne, was the Marché des Enfants-Rouges (opened in 1628), which at one time belonged to the Cassinis.[1] It took its name from a hospital nearby. César-François Cassini de Thury lived for a time at No. 8, Rue Simon-le-Franc, an ancient street known to have existed in 1200, and taking its name from Simon Franque, a wealthy merchant of the time. It is a little south of the Rue Michel-le-Comte which is mentioned later. The Rue de Bretagne crosses the Rue des Archives, a modern thoroughfare which includes the ancient Rue du Grand-Chantier on which, at No. 5 (old numbering), Jean Picard [2] lived.

Returning for a moment to the Ile de la Cité, facing the Palais de Justice is Notre Dame, near the north tower of which stood, until 1748, the small church of Saint-Jean-le-Rond [3] on the steps of which Mme. de Tencin's[4] child had been left on a winter's night and had been saved to become the great d'Alembert. The church has long since disappeared, and Mme. de Tencin has long since been dust under the pavement of Saint-Eustache, across the street from Les Halles Centrales. North of the site of Saint-Jean-le-Rond is the Rue des Ursins on which (at No. 9) stood, seven centuries ago, the cell of Abailard [5] who may possibly be ranked among the mathematicians for his study of Euclid in connection with his lectures on logic. He is buried with Héloïse by his side, in Père Lachaise. Two minutes' walk from here, across the little bridge to the Ile Saint-Louis, is the Rue Budé which takes its name from the mathematician, already mentioned, who prevailed upon Francis I to found the Collège de France. Returning to the right bank, a short distance south of Les Halles Centrales is the short Rue des Déchargeurs,[6] off which runs the Rue du Plat-d'Étain in which, at No. 1, was a cabaret in d'Alembert's time which he is said to have frequented. He was also a welcome guest at the salon of Mme. Geoffin, "le Ministre de la Société," as she was called, who lived not far away at No. 374, Rue Saint-Honoré. This street ends near Les Halles, and not far to the northeast lies the little Rue Michel-le-Comte where d'Alembert spent his early years in the home of a glazier. There is an old saying, "Ça fait la rue Michel," meaning "Ça fait le compte," a pun on "Michel-le-Comte." The Rue Michel-le-Comte terminates in the Rue du Temple which formerly, south of this point, was called the Rue Saint-Aroye. The Rue de Braque runs eastward from near this corner, and a letter was written

[1] One of the most remarkable families of mathematicians and astronomers. The leading members were Giovanni Domenico (1625–1712), who was known as Jean Dominique after going to Paris (1669); Jacques, his son (1677–1756); César-François de Thury (1714–1784), the son of Jacques; and Jacques Dominique de Thury (1742–1845), the son of César-François. They were all astronomers royal of France.

[2] 1620–1682, a pupil of Gassendi's, and known chiefly for his work in geodesy and astronomy.

[3] Like the ancient Saint-Denis-du-Pas, it stood within the close (cloister, cloître) of Notre Dame.

[4] The father, General Destouches, died in 1726.

[5] Or Abélard.

[6] It dates from the early 14th century and takes its name from the stevedores who lived there.

[7] Alexis Claude Clairaut, born at Paris in 1713; died at Paris in 1765, well known for his work in higher plane curves, dynamics, and astronomy. He became a member of the Académie des sciences at the age of 18.

by Clairaut[7] from "rue St Aroye vis á vis la rue de braque" on November 24, 1764. He died, very likely there, the following May.

To the east of Les Halles there runs the Boulevard Sébastopol, and a little to the north is the Square des Arts et Métiers, with the Conservatoire facing upon it. In this the mathematician and physicist will find a storehouse of material, including a model of Foucault's pendulum and a collection of calculating machines dating from those of Pascal to the period just preceding the recent great development of these nerve- and labor-saving devices.

Across the street to the south lies the Église Saint-Nicolas-des-Champs, the Temple de l'Hymen of the Revolution. Here are buried the Guillaume Budé already mentioned, and also Pierre Gassendi,[1] who was professor of mathematics at the Collège Royal (1645) and author of various works, chiefly on astronomy. The south side of the Conservatoire faces upon the modern (1851) Rue Réaumur. In the part to the west of the Boulevard de Sébastopol was the Rue Thévenot where, at No. 23 (old numbering), there lived, in 1808, Charles Bossut,[2] well known for his history of mathematics[3] and for his various textbooks and his monographs on geometry.

In the same general vicinity, on the Rue de Rivoli at the Square Saint-Jacques, there stands the Tour Saint-Jacques, dating from 1521. It is all that remains of the old gothic church of Saint-Jacques-la-Boucherie, built upon the site of a chapel of the 8th century, called the Capella in Carnificeria (we might, in America, say the church in the stockyards) and dedicated to Saint James (Saint Jacques). The church was taken down in the year 5 of the Republic (1796), and in 1856 the tower was put in its present condition, with a statue of Pascal between the arches. It was on this tower that Pascal made his experiments with a barometer before making them at Puy-de-Dôme (1648). The story that these were made in the tower of the church of Saint-Jacques-du-Haut-Pas, in the Rue Saint-Jacques, is unfounded. This latter church has some mathematical interest, however, for it is here that the geometer Lahire[4] is buried.

A little east of the Tour Saint-Jacques and north of the Rue de Rivoli is the Rue Sainte-Croix-de-la-Bretonnerie, formerly (1228) the Champs-des-Bretons. At No. 16 is the hôtel of Lalande, whom Montucla, in one of his letters, sarcastically called "the divine Lalande," and who edited the second edition of his Histoire des Mathématiques (1799–1802).

A short walk to the north along the Rue du Temple brings one to the Rue de Rambuteau, part of which, where now stands the Archives, was formerly called the Rue de Paradis (not to be confused with another street by the same name southwest of the Gare Saint-Lazare), and it was at No. 1 of this street that

[1] Born in 1592; died at Paris in 1655.

[2] Born near Lyons in 1730; died at Paris in 1814. A letter written by Delambre on February 9, 1808, was sent to him at this address.

[3] *Essai sur l'histoire générale des mathématiques*, Paris, 2 vols., 1802. English translation, London, 1803.

[4] Philippe de Lahire, born at Paris in 1640; died at Paris in 1718. The name often appears as La Hire.

Delambre[1] lived in 1802, and at No. 16 in 1807. Between this street and the Rue Sainte-Croix-de-la-Bretonnerie is the Rue des Guillemites, part of which was called at one time the Rue des Singes, and here the father of the unfortunate Bailly[2] had his home and here no doubt his gifted son was born.

IV. Street Names.

The custom of naming the streets after the great men of a city or of the world is not peculiar to Paris; it is seen in all continental countries, but it is more evident in Paris than in other places owing to the size of the city and its long history as an intellectual center.

Among the streets that bear the names of mathematicians there is, in the vicinity of the Boulevard Sébastopol, the Rue Borda, opened in 1816 and named in honor of Jean Charles Borda,[3] who was distinguished not only in mathematics but also in physics and navigation. In the same general region there is a street that was (from 1780 until 1884) known as the Rue de Breteuil, so called in honor of Émilie de Breteuil, Marquise du Châtelet, but now called the Rue du Général-Morin. A short distance away is the Rue Bailly, the name of which may suggest the mathematician elsewhere mentioned in this article, but which is really due to the bailliage of Saint-Martin-des-Champs to which church the property formerly belonged. Similarly, the Rue Dupin, which branches off from the Rue du Cherche-Midi near the Boulevard du Montparnasse, is not named after the mathematician, Charles, but after his brother André Marie Jean Jacques (1783–1865), the lawyer.

In the newer parts of the city little attention has been paid to the naming of streets with respect to their proximity to schools of any kind. In the northwest section, a short distance south of the Boulevard des Batignolles, is the Rue Bernoulli,[4] named after the mathematician who, to use the misprint in the Marquis de Rochegude's work, "découvrit le calcul expérimental!" [5]

South of the Arc de Triomphe de l'Étoile, in the relatively new section, are the Rue Euler,[6] the Rue Newton,[7] and the Rue Galilée,[8] all coming together on the Avenue Marceau. The Rue Keppler[9] is in the same vicinity. The Rue Galilée ends at the west in the Avenue Kléber, across which, extending from the Place Victor Hugo, is the Rue Copernic,[10] known before 1856 as the Rue des Bassins. North of the Place Victor Hugo runs the Rue Léonard de Vinci,[11]

[1] Jean-Baptiste-Joseph Delambre, born at Amiens in 1749; died at Paris in 1822. His greatest work was in geodesy and astronomy.

[2] See page 107.

[3] Born in 1733; died at Paris in 1799.

[4] Jean Bernoulli (1667–1748).

[5] For "exponential."

[6] Léonard Euler, born at Basel in 1707; died at Petrograd (Petersburg) in 1783.

[7] Sir Isaac Newton, born at Woolsthorpe in 1642; died at Kensington in 1727.

[8] Galileo Galilei, born at Pisa in 1564; died at Florence in 1642.

[9] Johann Kepler, born at Weil in Würtemberg in 1571; died at Ratisbon in 1630.

[10] Nicolaus Copernicus (Coppernicus), the astronomer (1473–1543), who also wrote on trigonometry.

[11] Leonardo da Vinci (1452–1519).

named for one who would have been a great mathematician if he had not been greater in nearly every other line that his genius touched. Just west of the Gare du Nord is the Rue Condorcet, with its little cité Condorcet at No. 27. At No. 21, Rue Chanzy, which branches off from the Boulevard Voltaire at No. 212, there lived Jean Macé (1815–1894) who, perhaps, deserves to be included in our list because of his little *L'Arithmétique du Grand-Papa* (1862). Near the Gare de Lyon and running south from the Avenue Daumesnil, is the short Rue Charles-Bossut [1] and the Rue Michel-Chasles,[2] the latter being part of the land occupied by the old prison de Mazas. The Rue Abel [3] was opened at about the same time (1901) and in the same territory. Some distance to the west, and near the Métro station of Place de Vaugirard, is the Rue Gerbert, leading from the church Saint-Lambert to the Rue de Vaugirard and bearing the name of the most learned mathematician of his day,[4]—a day in which mathematics, however, was near its low ebb.

In the Gobelin region the Boulevard Arago[5] was opened in 1859, and at the western section is the observatory, near which is the Rue Méchain.[6] Arago was director of the observatory and died there in 1853. In the court is a statue of Leverrier,[7] by Chapu. The observatory grounds are bounded on the north by Rue Cassini,[8] known in the 17th and 18th centuries as Rue des Deux-Anges, des Deux-Maillets, and Rue Maillot. Its present name dates from 1790. The Cassini house was No. 2. The boulevard north of the one which bears Arago's name is Boulevard de Port Royal. At No. 119 was the ancient Abbaye des Religieuses de Port-Royal which, in the 17th century, served as a retreat for many of the learned world, including, as is well known, Blaise Pascal.

North of the cemetery of Montparnasse are the Rue Delambre[9] and the Rue Huyghens.[10] South of the cemetery are the Rue Fermat,[11] the Rue Gassendi, and the Rue Lalande. Still farther south, and branching out from the Avenue d'Orleans, are the Rue Bézout and the Rue Sophie-Germain.

In the Quartier Latin there is the Rue Laplace,[12] so called since 1864, having been called the Rue de l'Allemanier as early as 1300. It lies north of the Panthéon.

[1] See page 168. The street received its name in 1873.

[2] Born at Épernon in 1793; died at Paris in 1880. He is known chiefly for his *Aperçu historique*, Paris, 1837, and his work on higher geometry (1852). The street was opened in 1902.

[3] Niels Henrik Abel, born at Findöe, Norway, in 1802; died at Arendal in 1829. Known chiefly for his work on elliptic functions and higher algebra.

[4] He became Pope Sylvester II in 999 and died in 1003.

[5] Dominique François Jean Arago (1786–1853), the astronomer.

[6] Pierre-François-André Méchain, born at Laon in 1744; died at Castellon de la Plana in 1804. He was one of the chief geodesists who worked on the base of the metric system.

[7] Leverrier is buried in the nearby cemetery of Montparnasse.

[8] See this MONTHLY, *1921*, 123, 369.

[9] The street was opened in 1839.

[10] French spelling for Huygens. Christian Huygens, born at The Hague in 1629; died at The Hague in 1695. Known as one of the greatest physicists and geometers of his time.

[11] Pierre de Fermat, born at Beaumont de Lomagne, *c.* 1608; died at Castres or Toulouse in 1665. Known for his work on theory of numbers and analytic geometry.

[12] Pierre-Simon, Marquis de Laplace, born at Beaumont-en-Auge in 1749; died at Paris in 1827. He wrote not only on astronomy but on probability and the higher calculus.

A little to the east is the Rue Descartes, known in the 13th century as the Rue Bordel (Bourdeille). As stated on page 174, Descartes was finally buried in Saint-Germain-des-Prés, this being recorded on a mural slab in the first chapel on the south after passing the sacristy. On the Rue Descartes stands the École Polytechnique[1] on the site of the ancient College de Navarre,[2] a school with which have been connected some of the greatest names in the history of French mathematics during the last century and a quarter. Back of the school runs the Rue Monge, named (1864) in honor of the great geometer. It is one of the most prominent of the many streets which were cut through ancient Paris in the 19th century and adds a posthumous glory to the memory of one who rose to great prominence but was allowed by his countrymen to die in poverty. The Square Monge lies next to the École Polytechnique where he worked for several years. There is also a Place Monge, and a little further south the Rue Pestalozzi[3] opens to the west.

Parallel to Boulevard Saint-Michel (the "Boul Mich" of the student jargon), and to the east, there runs the Rue Saint-Jacques, past the Sorbonne. From this there branches off to the west the short Rue Malebranche,[4] named in honor of one who was chiefly a philosopher but nominally a mathematician.

South of the Luxembourg gardens and running from the winding Rue Notre-Dame-des-Champs to the Rue d'Assas, is the Rue Leverrier, named after the scholar who applied mathematics to the determination of the position of Neptune. If one is visiting this section he may walk along the Rue Notre-Dame-des-Champs to No. 17, and see the house where Ampère[5] lived for a time. He may also walk over to the Rue d'Assas and see, at No. 28, the site of the house[6] in which Foucault died and in which he made some of his preliminary experiments on the pendulum.

Cutting the Boulevard Arago and running through to the Avenue des Gobelins is the Rue Pascal, opened in 1827.

In the northwestern part of the city, west of the cemetery of Montmartre and ending in the Avenue de Clichy, is the Rue Clairaut, before 1867 known as the Rue Sainte-Thérèse. The next street to the south, and a much longer one, is the Rue Legendre, which is cut by the Passage Legendre, formerly (1867–1877) known as the Passage Saint-Paul. Rue Legendre is cut farther to the west by the Rue Lemercier of which the southern part bears the name of Biot,[7] opened in 1850 but receiving its present name in 1864. Still farther to the west is the

[1] Founded in 1794.

[2] Founded in 1304 by Jeanne de Navarre, wife of Philippe le Bel.

[3] Johann Heinrich Pestalozzi, born at Zürich, 1746; died at Brugg, 1827. Known for his reform in elementary teaching of arithmetic and as the teacher of Steiner.

[4] Nicolas Malebranche (1638–1715). His mathematical work was chiefly in connection with astronomy.

[5] André Marie Ampère, born at Lyons in 1775; died at Marseilles in 1836. He was professor of mathematics at the École Polytechnique as well as professor of physics at the Collège de France. He wrote numerous monographs upon the calculus, higher plane curves, and probability, as well as upon physical questions.

[6] The present building is the École libre de Jeunes Filles.

[7] Jean-Baptiste Biot (1774–1862), who applied mathematics to physics and astronomy.

Boulevard Malesherbes on which is the Lycée Carnot, known as the École Monge until 1875, and this is bounded on the north by the Rue Viète,[1] so called in honor of the greatest of the early French algebraists. At the northern end of the same boulevard is the Rue Nicolas-Chuquet,[2] named for one of the few mathematical scholars of France at the close of the 15th century. From about the middle of the boulevard there runs to the westward the Rue Ampère, with an array of modern houses. It is cut by the Avenue de Wagram, which in turn is cut by the Rue Poncelet [3] with its small Passage Poncelet. Five minutes' walk to the west will bring one to the Rue Torricelli,[4] and another five minutes leads to the Rue Vernier and the Rue Roger Bacon,[5] before 1907 known as the Rue Bacon from its former proprietor. A short walk to the south brings one to the Avenue de la Grande Armée, between the Place de l'Étoile and the Bois de Boulogne, from which, near the Porte de Neuilly, branches the Rue Denis-Poisson, which went by divers names before the present one was adopted (1907). From the Étoile there starts, just north of the Avenue de la Grande Armée, the Avenue Carnot [6] which was so named in 1880 in honor of the Carnot of the Revolution.

Not far to the east of the cemetery of Montmartre, in the maze of narrow streets of the section, is the Rue Francœur,[7] a name given to the street in 1875.

Passing over to the northwestern part of the city, there lies, a short walk to the north of the Parc des Buttes Chaumont, the Passage Barrême, named in memory of the author of a very successful textbook on arithmetic,[8] one which still makes his name synonymous with all sorts of tabular devices for ready computation. Still in the eastern section, but near the Père-Lachaise cemetery, is the Rue Ramus,[9] recalling the death of the philosopher-mathematician who perished at the massacre of Saint Bartholomew, the signal for which was given by the bells of Saint-Germain-l'Auxerrois, across the street from the Colonnade of the Louvre. Branching off from the Rue Ramus is a small Passage Ramus, and a short walk to the east brings one to the Rue Vitruvius, calling to mind one

[1] François Viète, commonly known as Vieta, born at Fontenay-le-Comte in 1540; died at Paris in 1603.

[2] A native of Paris. He wrote the *Triparty en la Science des Nombres* at Lyons in 1484, but it was not printed until the 19th century.

[3] Jean-Victor Poncelet, born at Metz in 1788; died at Paris in 1867. He is known for his great work in projective geometry.

[4] Evangelista Torricelli, born in or near Faenza in 1608; died at Florence in 1647. Known not only as a physicist but for his contributions to the study of higher plane curves.

[5] Roger Bacon (1214–1294), the foremost scientist of his time, familiar with much of the older mathematics. He had an unusual knowledge of the latter science considering the time in which he lived.

[6] Lazare-Nicolas-Marguerite Carnot, born at Nolay in 1753; died at Magdeburg in 1823. A great military leader as well as a great geometer.

[7] See this MONTHLY, *1921*, 254.

[8] François Barrême, a native of Lyons. He died at Paris in 1703. His *Arithmétique* (Paris, 1677) went through many editions.

[9] Pierre de la Ramée, better known as Peter Ramus, born at Cust in 1515; died in 1572.

of the few Romans who had any appreciation of even the lower forms of mathematics.[1]

V. School Names, Statues, Tombs.

Paris has not only honored many of her mathematicians by the names of streets, but she has impressed upon her children the achievements of her learned citizens by occasionally giving such names to her schools. Thus we have the École Sophie-Germain, a higher primary school for girls at No. 9, Rue de Jouy, the street in which Richelieu was probably born and which takes its name from an abbey of the 13th century which stood at Nos. 13, 15, and 17. The Lycée Saint-Louis, on the Boulevard Saint-Michel, was at one time (1848) called the Lycée Monge, but the name was short-lived. The Lycée Condorcet is at No. 61, Rue d'Amsterdam, in the northwest section. The Lycée Voltaire (1890) is in the Avenue de la République, which leads out from the Place de la République. Besides these there are at No. 10, Boulevard Lannes, the École Pascal, and near the Place de la Nation the École Municipale Arago.

Paris has also many busts and statues and bas-reliefs of those who have made the science, some of relatively little importance, like the *médaillon* of Francœur on the building of the Société pour l'Instruction Élémentaire at No. 6, Rue du Fouarre, the ancient Rue des Escoliers of the 12th century, where several colleges were established in that period. Part of the street is now (since 1887) the Rue Lagrange. Budé [2] better deserves a statue, and one by Bourgeois (1882) stands in the Cours d'honneur of the Collège de France. Here are also the busts of Gassendi, Ramus, Lalande, Ampère, Chasles, and many others of scientific prominence.

On the chief façade of the Sorbonne is a statue representing mathematics and in the building are statues of Archimedes, Descartes, and Pascal. The garden of the Luxembourg has a statue of Bailly by the sculptor André, but it would be quite uninteresting to attempt anything like a complete list of efforts at portraiture of this kind to be found in the city. In the line of paintings, the most interesting portrait is the well-known one of Descartes, by Frans Hals, in the long gallery of the Louvre.

If the wanderer wishes to make a pilgrimage to the last resting-places of the mathematicians of Paris, his steps will naturally turn first to Père-Lachaise. Here he will do well to proceed by the Entrée principale up the main route, passing the graves of Arago (d. 1853) on the right (with a bust by David d'Angers) and of Poinsot (d. 1859) on the left, and then branching off to the Avenue Latérale du Sud to see the resting-place of Delambre (d. 1822). He may then proceed to the chapel, turn to the right, and pass on to the Carrefour du Rond, where he will find the graves of Monge (d. 1818), Hachette (d. 1834), and Fourier (d. 1830). A little farther along the Avenue des Acacias he will find the burial places of Chasles (d. 1881) and Comte (d. 1857). A few others are buried in different

[1] His great work on architecture was written between 20 and 14 B.C. He was well trained in the engineering of the time and in the ancient science of optics.

[2] See page 166.

parts of the cemetery, but the graves of many of the older group are to be found in less prominent cemeteries o. in the ancient churches. Louis Étienne Lefébure de Fourcy[1] (1787–1869), for example, is buried in the cemetery Montparnasse, as are also Poincaré (d. 1912) and Joseph Bertrand (d. 1900). Pascal is buried in Saint-Etienne-du-Mont and Descartes in Saint-Germain-des-Prés.[2] There is a monument to Maupertuis (d. 1759) in Saint-Roch, on the Rue Saint-Honoré, in the usual ultra-dramatic style of the times, and he was finally buried here.

Such are a few of the interesting historical spots of a city which has exerted a remarkable influence upon mathematics, pure and applied, in the last four centuries, and which is certain to have an equally remarkable influence in the centuries to come.

SOME UNSOLVED PROBLEMS IN SOLID GEOMETRY.[3]

By JULIAN LOWELL COOLIDGE, Harvard University.

One of the delightful uncertainties connected with mathematical research arises from doubt as to the life of this or that part of the subject. We have no vital statistics bearing on the longevity of different mathematical topics. No man can say at what moment any particular branch of the science may wither and die for lack of new life and growth. A process of senility seems to have set in in algebra, in the elementary theory of numbers, and in other parts of our science. One elementary branch alone seems exempt from this general law of decay, elementary geometry. For some reason this ancient branch of learning seems to bear in itself an inexhaustible fountain of youth. It is just a hundred years since Feuerbach first published his classic theorem about the nine-point circle, eleven years ago a Japanese mathematician published eleven new proofs.[4] The Brocard figures were discovered some fifty years ago, the number of published articles dealing with them quickly ran up towards a thousand.

The great majority of the best theorems of elementary geometry are, and always will be, "in plano." This is partly owing to the greater inherent simplicity of two dimensional figures, partly to the equality of all angles inscribed in the same circular arc, a property which has no analogue in space. Yet I am personally persuaded that there are a lot of simple and elegant theorems in solid geometry, waiting for discovery by skillful and painstaking geometers. Who could ask for a better theorem than that which tells us that if a sphere be inscribed in a tetrahedron, the three angles subtended at each point of contact, by the three edges coplanar with that point, are the same in every case. Yet this theorem, so far as we know, was first found in 1897, and did not come to general notice till mentioned by Franz Meyer at the International Mathematical Congress at Heidelberg in 1904. I wish that I knew the way through the wood

[1] He wrote several works on algebra and geometry.

[2] See this MONTHLY, *1921*, 62.

[3] Read before the Mathematical Association of America at Cambridge, December 29, 1922.

[4] V. Sawayama, "Nouvelles démonstrations d'un théorème relatif au cercle des neuf points," *L'Enseignement Mathématique*, vol. 13, 1911, pp. 31–49.

to the enchanted palace where numbers of such mathematical beauties are still asleep. Alas, I do not. In the present paper I must confine myself to the more prosaic task of pointing out various directions in which I believe interesting mathematical truths may lie, hoping that some geometers whose skill and patience exceed my own will go boldly ahead to discover them.

There are a large number of so-called notable points connected with the triangle. Every high-school pupil learns about the center of the circumscribed circle, where the perpendicular bisectors of the sides meet, the center of the inscribed circle where the bisectors of the angles meet, the center of gravity which lies on all the medians, and the orthocenter on the altitudes. Further study will lead him to Ceva's theorem giving the conditions for concurrence of lines through the vertices of a triangle, and if he take up the modern geometry of the triangle, he will hear about the Brocard points, the symmedian points, Miquel's point, Tarry's point "et hoc genus omne." How many of these points have their counterparts in the geometry of the tetrahedron?

The first steps in the search for such points are encouraging. The perpendicular bisectors of the edges of a tetrahedron meet in the center of the circumscribed sphere, the bisectors of the dihedral angles meet in the center of the inscribed sphere, the lines to the centers of gravity of the faces meet in the center of gravity of the tetrahedron, it all looks very hopeful. But when we examine whether the altitudes be concurrent or not, the trouble begins. There is a very widespread ignorance on this simple matter. I once asked the combined mathematical department of a large New England college, and not one of them had the ghost of an idea. As a matter of fact these altitudes are not, usually, concurrent. If at the orthocenter of each face we erect a perpendicular to that face, it is easy to see that each of these perpendiculars intersects three altitudes and is parallel to the fourth. We have two sets of four lines, all lines of one set intersecting all of the other in finite or infinite points, yet neither set are all parallel to one plane. This shows that they are generators of the same hyperboloid. It is true that the center of this hyperboloid makes a pretty good notable point,[1] but the first natural reaction is to feel discouraged from searching further. Yet such pessimism would, perhaps, be immature. Here is a ground for hope:

Suppose that we connect a point in the plane of a triangle, not on the circumscribed circle, with the three vertices, and reflect the three connecting lines in the bisectors of the angles; the reflected lines will also be concurrent. The two points of concurrence, which are said to be "isogonally conjugate," have a reciprocal relation, they are the two foci of a conic tangent to the lines of the sides of the triangle, and if we take a system of homogeneous trilinear coördinates, where a point is represented by numbers proportional to its distances from three non-concurrent lines, the coördinates of one point are proportional to the reciprocals of the coördinates of the other. Lastly if we drop perpendiculars from the two points on the three lines, the feet of these are six points on one circle, called the "pedal circle" of the two points.

[1] C. Intrigila, "Sul tetraedro," *Rendiconti R. Accademia delle Scienze di Napoli*, vol. 22, 1883, pp. 69–95.

Now these relations, contrary to what one would naturally fear, hold unaltered in three dimensions. Each point, not on a certain cubic surface, has a single isogonal conjugate with regard to a tetrahedron, and the two have the same pedal sphere, the tetrahedral coördinates of the one are the reciprocals of those of the other. These facts suggest the possibility of further progress. In the plane the center of the circumscribed circle and the orthocenter are isogonally conjugate, as are the center of gravity and the symmedian point.

PROBLEM 1. What are the geometrical properties of the points which are isogonally conjugate to such notable points of the tetrahedron as are already known?

In the plane, if a point trace a straight line, its isogonal conjugate will trace a conic through the vertices, and vice versa. When the line goes through the center of the circumscribed circle, the conic is an equilateral hyperbola whose center is on the nine-point circle. These facts, which do not correspond directly to anything in space, lead at once to the properties of the nine-point circle.

PROBLEM 2. What are the properties of the cubic surfaces and cubic space curves which correspond isogonally to planes and lines? When will these figures reduce to combinations of simpler figures?

We mentioned on a previous page that the altitudes of a tetrahedron are usually the generators of a hyperboloid, and so, not concurrent. There are, however, special cases where all go through one point. If two pairs of opposite edges have the property that their lines are mutually perpendicular in direction, the same is true of the third pair. The foot of each altitude is, in this case, the orthocenter of a face and the four altitudes are concurrent. We call such a tetrahedron an "orthogonal" one. If the faces be called 1, 2, 3, 4, and the angle between faces i and j be θ_{ij}, the condition for an orthogonal tetrahedron is immediately found to be

$$\cos\theta_{12}\cos\theta_{34} = \cos\theta_{13}\cos\theta_{24} = \cos\theta_{14}\cos\theta_{23}.$$

In the case of the orthogonal tetrahedron there are two twelve-point spheres, quite analogous to the nine-point circle.[1] We are led to:

PROBLEM 3. How are problems of isogonal conjugacy modified in the case of the orthogonal tetrahedron?

There is a popular dictum, which almost has the force of dogma, that all great inventions are essentially simple. Without going into the question of the limits of validity of this general statement, we may safely say that the most fruitful theorems in geometry are generally the simplest. The following is a case in point. Let us start with a triangle and mark a point on the line of each side. We now pass a circle through each vertex and the two marked points on lines through that vertex. Each circle and the opposite line form a cubic curve, and the three cubics have eight common points. They are, consequently, linearly dependent, and the three circles pass through a common point called the Miquel

[1] A. Transon, "Sur une question relative à la géométrie de l'espace," *Nouvelles Annales de Mathématiques*, series 2, vol. 2, 1863, pp. 138–143.

point.[1] If we invert the figure from a point either in the plane or outside of it we have a curvilinear triangle determined by three concurrent circles of a plane or sphere, with a point marked on each circle. The three circles, each through a vertex and two adjacent marked points, are concurrent. In fact we have two sets of reciprocally related concurrent circles. This relation may be immediately generalized in space if we go about it in the right manner. Let us start with a tetrahedron, and mark a point on the line of each edge, then pass a sphere through each vertex and the adjacent marked points. In each face we have the Miquel configuration, hence the given spheres are concurrent by threes in the planes of the faces. On each sphere we have the Miquel figure generalized by inversion, hence the four spheres go through a point.[2] At the same time the question may fairly be asked whether we have generalized Miquel's theorem in the most natural way. Would it not be more natural to mark a point in each plane? This leads to:

PROBLEM 4. In the plane of each face of a tetrahedron, a point is marked. What is the necessary and sufficient condition that the three spheres, each through a vertex and the marked points in the adjacent planes, should be concurrent?

This may also be restated in another form, more favorable to analytic handling:

PROBLEM 5. Given two sets of points, $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$. If the four spheres $A_iB_jB_kB_l$ be concurrent, when will the spheres $B_iA_jA_kA_l$ also be concurrent?

Returning to the plane, a very good way to mark points on three given lines is by their intersections with a fourth line. We are thus led to the theorem that the circles circumscribing the four triangles formed by sets of three out of four given lines are concurrent; the point of concurrence being, in fact, the focus of the parabola which touches the lines. Let us say that this point is "associated" with the four lines. If five lines be given, each set of four will be associated with a point, and the five points lie on a circle, associated with the five lines. If six lines be given, each set of five are associated with a circle, and the six circles are concurrent in a point associated with the six lines. Keeping on in this way we find that each even system of lines will be associated with a point, and each odd set with a circle in such a way that the point associated with $2n$ lines lies on the circle associated with each sub-set of $2n - 1$, and the circle associated with $2n - 1$ contains the point associated with each sub-set of $2n - 2$. This figure is called a "Clifford Chain," after the first of some three or four independent discoverers.[3] Other such chains have been found by Grace,[4] Pesci[5] and the

[1] A. Miquel, "Sur quelques questions relatives à la théorie des courbes," *Journal de Mathématiques pures et appliquées* (Liouville), vol. 3, 1838, pp. 202–208.

[2] S. Roberts, "On certain tetrahedra specially related to four spheres meeting in a point," *Proceedings of the London Mathematical Society*, vol. 12, 1880–1881, pp. 117–120.

[3] W. K. Clifford, "Synthetic proof of Miquel's theorem," *Oxford, Cambridge and Dublin Messenger of Mathematics*, vol. 5, 1870, pp. 124–141.

[4] J. H. Grace, "Circles, spheres and linear complexes," *Transactions of the Cambridge Philosophical Society*, vol. 16, 1897–1898, pp. 153–190.

[5] G. Pesci, "Dei circoli circoscritti ai triangoli etc.," *Periodico di Matematica*, vol. 5, 1896, pp. 120–127.

present author.[1] It seems self evident that there must be analogues in space to many of these chains, although, so far, only the last seems to have been found.[2] This is curious. Roberts' theorem seems to suggest natural extension. If we cut a tetrahedron by a plane, we mark a point very nicely on each edge line, and the spheres circumscribing the four tetrahedra, each having faces in three of the given planes, while its remaining face is in the secant plane, are concurrent. They are the spheres circumscribing four of the five tetrahedra determined by the five planes. But when we take the five tetrahedra and five circumscribing spheres they are not all concurrent in one point, but are concurrent by fours in five different points, one in each plane. It is not certain whether these five points are cospherical or not, I have the impression that they are not, but can not say why. At any rate there is very little known about possible extensions.

PROBLEM 6. Are there any three-dimensional analogues to the chains of Clifford, Grace and Pesci?

That master of geometrical craft, Jakob Steiner, devoted much time and attention to the following problem. Suppose that we have two circles, one surrounding the other. Can we construct a chain of circles, finite in number, each of which is tangent not only to the given circles but its two next neighbors in the chain? The problem is easily handled if we invert our two circles into concentric ones. It then appears that if we can construct one such chain of circles, we may construct an infinite number of them. Returning to the general case, let the line of centers cut the first circle in A_1B_1 and the second in A_2B_2, so that the two A's lie on one side, and the two B's on the other. Construct the two circles whose diameters are A_1B_2 and A_2B_1. If their angle of intersection be expressible in the form $2\pi m/n$ there will be a chain of n circles making m circuits.

If we generalize in space in such a way as to seek a chain of spheres tangent to three given spheres there is nothing essentially new, for it is evident that the problem of finding spheres tangent to two concentric spheres and to some third sphere goes right back to the two-dimensional case.[3] Suppose, however, we have only two given spheres, we then get:

PROBLEM 7. Can there be such a system of spheres tangent to two given spheres that each sphere is tangent to at least 3 others of the system?

Steiner's ring problem is but an example of a common type of problem to which the name of "closure" has been attached. Numbers of geometers, following Poncelet, have looked into the question of what relations must exist between two conics, if a polygon of n sides can be inscribed to the one and circumscribed to the other. This suggests

PROBLEM 8. What relation must exist between two spheres in order that

[1] J. L. Coolidge, "Some circles associated with concyclic points," *Annals of Mathematics*, series 2, vol. 12, 1910–1911, pp. 39–44. See article by F. V. Morley, "Note on the incenters of a quadrilateral," this MONTHLY, *1920*, 252–255.

[2] C. Intrigila, *loc. cit.*, pp. 78–79.

[3] K. Th. Vahlen, "Ueber Steiner'sche Kugelketten," *Zeitschrift für Mathematik und Physik*, vol. 41, 1896, pp. 153–160.

it may be possible to inscribe a tetrahedron in one which is circumscribed to the other?

It is hard to believe that this problem has never been solved, I can only affirm that I have never seen a solution, and there is no reference thereto in Simon's bibliography of elementary geometry in the nineteenth century.[1]

Steiner's contact problem leads naturally to an even more famous one of a slightly different nature. How can we put three circles inside a triangle in such a way that each shall touch two sides and the other two circles? This is known as Malfatti's problem,[2] and the literature bearing on it is depressingly extensive, one reason for its popularity being that Steiner published the following simple construction without proof: [3]

Bisect the angles of the triangle. Inscribe a circle in each triangle formed by one of the given sides and the adjacent bisectors. The bisectors are transverse common tangents to these circles, two by two. Draw the other transverse common tangents. The circles required each touch two of these last tangents and two of the sides.

The first demonstration was given thirty years later by Hart,[4] a priority which the Germans did not particularly relish.

Let us discuss the corresponding problem in three dimensions. It is evident that there are some tetrahedrons in which we can place four Malfatti spheres, for if we start with four spheres, each of which touches the other three externally, we can build a tangent tetrahedron outside. On the other hand it seems unlikely that four Malfatti spheres can be placed in every tetrahedron. There is just one triad of spheres mutually tangent, each of which touches the base of a tetrahedron and two lateral faces, inside. There are an infinite number of spheres tangent to the three lateral faces inside, but they have only one degree of freedom, so that we can hardly require one of them to touch the three spheres already found. It seems likely that two independent conditions must be imposed upon a tetrahedron if we are to find four Malfatti spheres. We find those conditions by using an identity apparently due to Frobenius.[5] Suppose that six spheres or planes are indicated by the numbers 1, 2, 3, 4, 5, 6. Let the angle between spheres i and j be indicated by θ_{ij}. Then

$$\begin{vmatrix} 1 & \cos\theta_{12} & \cos\theta_{13} & \cos\theta_{14} & \cos\theta_{15} & \cos\theta_{16} \\ \cos\theta_{21} & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cos\theta_{61} & \cdot & \cdot & \cdot & \cos\theta_{65} & 1 \end{vmatrix} = 0.$$

[1] M. Simon, *Über die Entwicklung der Elementar-Geometrie im XIX Jahrhundert*, Leipzig, 1906.

[2] *Memorie di matematica e di fisica della Societa Italiana delle Scienze Modena*, 1803. I have never been able to see this article.

[3] Steiner, "Einige geometrische Betrachtungen," *Journal für die reine und angewandte Mathematik* (Crelle), vol. 1, 1826, pp. 161–184 (see § IV, p. 178).

[4] A. S. Hart, "Geometrical investigation of Steiner's construction for Malfatti's problem," *Quarterly Journal of Mathematics*, vol. 1, 1857, pp. 219–221.

[5] G. Frobenius, "Anwendungen der Determinantentheorie auf die Geometrie des Maasses," *Journal für die reine und angewandte Mathematik* (Crelle), vol. 79, 1874, 185–247.

In the present case we may take the angle between each two spheres as π while the sphere i makes an angle φ_i with the like numbered plane and 0 with the other 3 planes. Taking the planes i and j and the four spheres, we have

$$\begin{vmatrix} 1 & \cos\theta_{ij} & \cos\varphi_i & 1 & 1 & 1 \\ \cos\theta_{ij} & 1 & 1 & \cos\varphi_j & 1 & 1 \\ \cos\varphi_i & 1 & 1 & -1 & -1 & -1 \\ 1 & \cos\varphi_j & -1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & 1 \end{vmatrix} = 0.$$

The development of this determinant requires care rather than skill. The resulting equation is

$$\cos\varphi_i \cos\varphi_j (\cos\varphi_i + \cos\varphi_j) + A_{ij} \cos\varphi_i \cos\varphi_j + B_{ij}(\cos\varphi_i + \cos\varphi_j) + C_{ij} = 0,$$

where A_{ij}, B_{ij}, C_{ij} are simple polynomials in $\cos\theta_{ij}$. There will be six such equations. Eliminating the four angles φ_i we must have two eliminants in the six angles θ_{ij}, which are probably independent.

PROBLEM 9. Is there any connection between these equations and the conditions for an orthogonal tetrahedron?

PROBLEM 10. When the conditions are fulfilled, how are the Malfatti spheres constructed?

LINEAR OPERATIONS AND GENERALIZED ELEMENTARY SYMMETRIC FUNCTIONS.

By ALBERT A. BENNETT, University of Texas.

Many of the simplest operations of mathematics are linear, and among these the linear homogeneous operations are the most interesting. When attention is confined to rational algebraic steps with a single variable x the most general linear homogeneous operation available is that of multiplying x by a constant. The Cauchy functional relation, $F(x + y) = F(x) + F(y)$, which features in this connection has also celebrated discontinuous solutions.

Without attempting to classify all linear operations, it may be of interest to point out five special illustrations of the general class of homogeneous linear operations. In each of these there are two sets of quantities, $[a]$ and $[F(a)]$, having elements, a, b, c, $\cdots$, and $F(a)$, $F(b)$, $F(c)$, $\cdots$, respectively. In each case, $F(a + b) = F(a) + F(b)$, addition being defined for both sets. The significance of the following discussion lies in the fact that it is applicable equally to all of these distinct cases.

While it is possible to discuss the merely additive properties of linear functions, the existence of a basis, finite or infinite, the nature of the roots of a linear equation, and so forth, we shall here be interested in the application of multiplication to linear homogeneous operations. We shall assume indeed that both for the

set $[a]$ and for the set $[F(a)]$ commutative associative multiplication is possible, this multiplication being distributive with respect to addition. The five illustrations or examples of the general theory are as follows:

1. $[a]$ denotes the set of expressions, $a = (x_1, x_2, \cdots, x_n)$, $b = (y_1, y_2, \cdots, y_n)$, $\cdots$, for a fixed n. $F(a)$ denotes $\sum_{i=1}^{n} x_i$. By the product ab will be meant the element of $[a]$ that may be written, $(x_1y_1, x_2y_2, \cdots, x_ny_n)$. The other sums and products are defined as is usual in algebra.

2. $[a]$ denotes the set of algebraic numbers of a given algebraic field of order, n. By $F(a)$ is meant the trace (*Spur*) of a, so that $F(a)$ is always a rational number. The sums and products are as usual in algebra.

3. $[a]$ denotes the set of matrices which are polynomials (with scalar coefficients) in a given non-singular square matrix of order n, with distinct invariant factors. $F(a)$ is the sum of the elements in the main diagonal of a. The product ab is the usual matric product, which in this case is commutative. The sum $a + b$ is the matric sum. The sum and product for elements of $[F(a)]$ is as in algebra.

4. $[a]$ denotes the set of finite subsets, repeated elements being permitted, of a given set of objects. $F(a)$ denotes the number of elements in the subset a, any element occurring exactly n times in a, yielding the number n in the count $F(a)$. In defining the sum $a + b$ each element that appears in either term appears in the sum, and the number of times that it occurs in the sum is the sum of the number of times that it appears in the given terms. By the product ab is meant the subset, possibly the null-set, containing only elements common to both factor sets, the number of times that any given element appears being the product of the number of times that it appears in the given factors. The sum and product for $[F(a)]$ are defined as in arithmetic.

5. $[a]$ denotes the set of subsets, finite or infinite, without repeated elements, of a given set of elements. $F(a)$ is the same set as a, but with different rules of combination. The sum $a + b$ is the arithmetic sum defined in 4, and $F(a) + F(b)$ coincides with it. The product ab is as in 4, but a new definition is given to the product $F(a)F(b)$. This is in fact defined as the set of all unordered pairs of elements, one each from $F(a)$ and from $F(b)$, respectively, any pair consisting of the same element twice, once each from $F(a)$ and from $F(b)$, counting merely as this element itself.

The sets $[a]$ and $[F(a)]$ and their rules of combination having been once settled upon, it is possible to introduce a sequence of functions of more than one argument, which functions play an important part in the theory of each of the illustrations and constitute the basis of the general theory. These will be denoted by $F_1, F_2, F_3, \cdots, F_n, \cdots$ where the subscript in each case identifies the function and also denotes the number of independent arguments which enter into its definition. Thus these are respectively of the form $F_1(a)$, $F_2(a, b)$, $F_3(a, b, c)$, $\cdots$, $F_n(a_1, a_2, \cdots, a_n)$. Each $F_n(a_1, a_2, \cdots, a_n)$ will be seen to be linear and homogeneous in each of the arguments, as a consequence of the defining relations that will be given presently. Furthermore it will be shown that each integral

rational function of the F's with arguments which are themselves integral rational functions of the independent arguments a, and which is such as to be linear and homogeneous in each of these independent arguments, is uniquely expressible as a linear combination of the F's.

The functions in the sequence $F_1, F_2, \cdots, F_n, \cdots$ are themselves linearly independent. The simplest method of defining the relations among their products is perhaps to give a sequence of equations, each introducing in turn a single one of these functions. On the basis of these, other relations can be derived which must be valid in all cases and not dependent upon the special interpretation of $[a]$ and $[F(a)]$. Incidentally the theory of the elementary symmetric functions of algebra follows by a specialization in illustration 1. The definitions adopted are as follows:

$$F_1(a) = F(a),$$
$$F_1(a)F_1(b) = F_1(ab) + F_2(a, b),$$
$$F_1(a)F_1(b)F_1(c) = F_1(abc) + F_2(ab, c) + F_2(ac, b) + F_2(bc, a) + F_3(a, b, c), \quad (1)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$F_1(a_1)F_1(a_2)\cdots F_1(a_n) = F_1(a_1a_2\cdots a_n) + \sum F_2 + \sum F_3 + \cdots + F_n(a_1, a_2, \cdots, a_n),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Here the sign $\sum$ in each case covers all distinct integral partitions of the n arguments, $a_1\, a_2, \cdots, a_n$, into products to the number required by the subscript indicated. It is now seen that each F, in turn, is symmetrical in its arguments. The summation sign covers a determinate but, except in the simplest cases, large numbers of terms. The expansion of the product $F_1(a)F_1(b)F_1(c)F_1(d)F_1(e)$ will amply illustrate the formation of these sums. For this case, $\sum F_2$ contains the following fifteen terms,

$$\begin{aligned} &F_2(abcd, e) + F_2(bcde, a) + F_2(acde, b) + F_2(abde, c) + F_2(abce, d) \\ &\quad + F_2(abc, de) + F_2(abd, ce) + F_2(acd, be) + F_2(bcd, ae) + F_2(abe, cd) \\ &\quad + F_2(ace, bd) + F_2(bce, ad) + F_2(ade, bc) + F_2(bde, ac) + F_2(cde, ab). \end{aligned}$$

On the other hand $\sum F_3$ contains twenty-five terms,

$$\begin{aligned} &F_3(abc, d, e) + F_3(abd, c, e) + F_3(acd, b, e) + F_3(bcd, a, e) + F_3(abe, c, d) \\ &\quad + F_3(ace, b, d) + F_3(bce, a, d) + F_3(ade, b, c) + F_3(bde, a, c) + F_3(cde, a, b) \\ &\quad + F_3(ab, cd, e) + F_3(ac, bd, e) + F_3(ad, bc, e) + F_3(ab, ce, d) + F_3(ac, be, d) \\ &\quad + F_3(ae, bc, d) + F_3(ab, de, c) + F_3(ad, be, c) + F_3(ae, bd, c) + F_3(ac, de, b) \\ &\quad + F_3(ad, ce, b) + F_3(ae, cd, b) + F_3(bc, de, a) + F_3(bd, ce, a) + F_3(be, cd, a). \end{aligned}$$

There are but ten terms in $\sum F_4$, namely,

$$\begin{aligned} &F_4(ab, c, d, e) + F_4(ac, b, d, e) + F_4(ad, b, c, e) + F_4(ae, b, c, d) + F_4(bc, a, d, e) \\ &\quad + F_4(bd, a, c, e) + F_4(be, a, c, d) + F_4(cd, a, b, e) + F_4(ce, a, b, d) + F_4(de, a, b, c). \end{aligned}$$

The defining equations (1) exhibit the products of F's when these are of the simplest sort, as linear expressions in F's. It remains to show that under proper conditions of homogeneity alone, as already mentioned, any product of F's is itself expressible as a linear combination of F's. Owing to the linearity of the

F's in the several arguments, no further discussion is needed to extend the results from mere products to general rational functions suitably homogeneous. It suffices to consider only products of two F's, since if the theorem is true for all such products, it will follow for products of more than two factors by mere repetition and summation. Consider therefore the product $F_m(A_1, A_2, \cdots, A_m)F_n(A_1, A_2, \cdots, A_n)$, where each A is either an a, or a product of a's. We shall first cast the equations (1) into another form,

$$F_n(a_1, a_2, \cdots, a_n) = F_1(a_1)F_1(a_2) \cdots F_1(a_n) - F_1 - \sum F_2 - \sum F_3 - \cdots - \sum F_{n-1}. \quad (2)$$

Making use of the form (2) for each of the two factors F_m and F_n, we obtain a sum of terms in which the subscript m is replaced by smaller integers, and likewise n is replaced with smaller integers, together with terms involving F_1 as a repeated factor. By repeated applications, the original product is eventually replaced by a sum of terms each containing exclusively repeated factors F_1. Finally, each of these products is evaluated as a linear expression by use of formula, (1). An indication of the result to be expected in simple cases may be inferred by reference to the product, $F_3(a, b, c)F_1(d)$ which reduces after the above mentioned operations have been performed to $F_3(ad, b, c) + F_3(bd, a, c) + F_3(cd, a, b) + F_4(a, b, c, d)$.

We are now prepared to interpret $F_2(a, b)$ in each of the five illustrative cases mentioned above. In 1, $F_2(a, b)$ denotes the sum of all products $x_i y_j$, $i \neq j$. In 2, $F_2(a, b)$ is always a rational number. It is a symmetric function of the pair of algebraic numbers a, b, but has no commonly current name, other than $S(a)S(b) - S(ab)$, where $S(x)$ is the trace of x. In 3, $F_2(a, b)$ is a symmetric scalar function of the pair of matrices, a, b, but enjoys no special name. In 4, $F_2(a, b)$ is a number dependent upon the number of elements which are common to the subsets a and b. If a contains n elements not necessarily distinct, $F(a)$ is n. If b contains m elements, $F(b)$ is m. If no element of a is also in b, $F(ab) = O$. If however a and b should coincide, but the elements of each should be distinct, then $m = n$, and $ab = a = b$, so that $F(ab) = n$, and $F_2(a, b) = n(n - 1)$. If a consists of a single element counted n times and b consists of the same element counted m times, $F(a)F(b) = F(ab)$, so that $F_2(a, b) = O$. In 5, the expression, $F_2(a, b)$ consists of the set of unordered pairs of distinct elements one each from a and from b.

The theory of the functions F_m with m distinct arguments is fundamental, but in most applications functions are used in which the arguments are not independent but are all powers of a single argument. In the most familiar cases, indeed, the arguments are all equal, so that F_1, F_2, F_3, etc., are determined by a single quantity a. Equations (1) become in this important case,

$$\begin{aligned} F_1(a) &= F(a), \\ F_1^2(a) &= F_1(a^2) + F_2(a, a), \\ F_1^3(a) &= F_1(a^3) + 3F_2(a^2, a) + F_3(a, a, a), \\ F_1^4(a) &= F_1(a^4) + 4F_2(a^3, a) + 6F_2(a^2, a^2) + 4F_3(a^2, a, a) + F_4(a, a, a, a). \end{aligned} \quad (3)$$

. . . . . .

It may be noted that from the same function $F_2(a, b)$ are obtained in the fourth of the above relations two distinct functions of the single argument a. To determine the number of times that a given function of several distinct arguments will appear in the equation for a given power of $F(a)$, say the mth, we must find the number of distinct partitions of which m is capable, where the number of parts for each partition is equal to the subscript of the function in question.

Making use of the specialization explained above, we are in the position to give further interpretations in the five illustrative cases. It will be convenient to refer occasionally to the expression,

$$x_n - F_1(a)x^{n-1} + (F_2(a, a)/2)x^{n-2} - \cdots + (-1)^n(F_n(a, a, a, \cdots, a)/n!). \quad (4)$$

In 1, where $a = (x_1, x_2, x_3, \cdots, x_n)$, $F_2(a, a)$ consists of the sum of products x_ix_j, $i \neq j$, and contains each of these products twice, namely once in the form x_ix_j and once in the form x_jx_i. Thus $F_2(a, a)/2$ is the elementary symmetric function of the second order. More generally, $F_m(a, a, \cdots, a)/m!$ is the mth elementary symmetric function of x's in a. In this illustration, the expression (4) is a polynomial in x whose roots are the elements of a. For this reason the expression $F_m(a, a, \cdots, a)/m!$ may be regarded in general as a generalized elementary symmetric function.

In 2, a is an algebraic number in a field of order n. This does not require that a is itself of the nth order, but in any case there is a unique polynomial of the nth order defined having a as a root, and which is either irreducible in the domain of rational numbers or else is merely a power of an irreducible polynomial. This polynomial will be expressible in the form (4). Here $F_1(a)$ is the trace and $F_n(a, a, \cdots, a)/n!$ is the norm of a. The intermediate coefficients are likewise rational numbers featuring in the theory.

In 3, the well-known "characteristic equation" furnishes the first elements of the theory. The characteristic equation does not completely define the matrix a, if not all the roots of this equation are distinct, there being additional arithmetic invariants in this case. If the roots be distinct, however, no further invariants exist for the transformations usually considered in the algebraic or geometric applications. The characteristic equation is obtained by putting (4) equal to zero. Thus the coefficient $F(a, a, \cdots, a)/n!$ is merely the determinant of a.

In 4, $F_m(a, a, \cdots, a)/m!$ is a non-negative integer for each choice of m. The term contributed to F_m by any repeated element of a is $n!/(n - m)!$, where n is the number of times that the given element appears in a. This functional symbol is of course to be understood to denote 0, whenever m exceeds n, and to denote $n!$ whenever $m = n$. Since the binomial coefficient $n!/[m!(n - m)!]$ is an integer it follows that in each case, F_n is divisible integrally by $m!$ and vanishes if no element of a is counted as many as m times.

If in 5, a be thought of as in some sense a "linear" set, $F_2(a, a)$ will be twice a triangular set, $F_3(a, a, a)$ will be six times a tetrahedral set, and so forth, $F_n(a, a, \cdots, a)$ being $n!$ times a set which is a simplex of n dimensions. These simplexes constitute a generalization of the binomial coefficients.

Given the problem of expressing the symmetric function of more than four variables represented by $\sum x_1^2 x_2^2 x_3$, in terms of the elementary symmetric functions of these variables, a possible method of solution, in case that the number of variables is not excessive, is actually to expand the indicated summation as the first step. Since the form of the result is the same no matter what the number of variables so long merely as there are enough to give all of the indicated sums a meaning, it is clear that there is no necessity of making the expansion. The solution can be effected by use of the relations (1) and relations derived therefrom alone. The problem can therefore be put into a more general form. We can inquire into the representation of the expression $F_3(a^2, a^2, a)/2$ [1] as a sum of products of the elementary functions $F_1(a)$, $F_2(a, a)$, $F_3(a, a, a)$, etc., alone. The proof that a solution is always possible may be made in the usual manner provided that care be taken in the wording of the proof. To simplify the notation, it is usually convenient to adopt the nomenclature current in the theory of the elementary symmetric functions of algebra, and write $F_n(a, a, \cdots, a)/n!$ in the condensed form $E_n(a)$. Using this notation, we have as the result of a few steps, $F_3(a^2, a^2, a)/2 = E_2E_3 - 3E_1E_4 + 5E_5$. This result may be interpreted in connection with each of the illustrations already considered. It is of particular interest to consider 5. In this case since $a^2 = a$, it follows that $F_3(a^2, a^2, a)$ is the same as $F_3(a, a, a)$ which is itself equal to $6E_3(a)$. Thus for this particular illustration, $3E_1E_4 + 3E_3 = E_2E_3 + 5E_5$. When, in particular, the set a is finite and consists of n elements, the count of the number of elements gives a relation among the binomial coefficients, but the theorem is one applicable to any sets. To take a simpler example, from the general relation $F_2(a^2, a) = E_1E_2 - 3E_3$, we have for 5, $2E_2 = E_1E_2 - 3E_3$, or in other order, $E_1E_2 = 3E_3 + 2E_2$. Geometrically interpreted, this means that the triangular prism obtained from a given linear set, consists of two triangular interfaces and three tetrahedra, a relation familiar in solid geometry, but here valid for a general "linear" set.

THE AREA OF RULED SURFACES BY VECTORS.

By J. B. REYNOLDS, Lehigh University.

When a surface may be considered as generated by the movement of a straight line, we may think of the line as a vector and thus easily lead up to an expression for the area generated by the moving line.

Let $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ be vectors from the origin of vectors to the ends of the generating vector $\boldsymbol{l}$, as shown in figure 1. If then $\boldsymbol{r}$ is the vector to any point on the surface, we have $\boldsymbol{r} = \boldsymbol{r}_1 + s\boldsymbol{l}$, in which s is a scalar quantity. From this

$$d\boldsymbol{r} = d\boldsymbol{r}_1 + s d\boldsymbol{l} + \boldsymbol{l} ds,$$

[1] The occasion for the factor 2 is that in $F_3(a^2, a^2, a)$, itself, the following terms for example would all appear as though distinct, $x_1^2x_2^2x_3$, $x_2^2x_1^2x_3$, $x_2^2x_3^2x_1$, $x_3^2x_2^2x_1$, $x_3^2x_1^2x_2$, $x_1^2x_3^2x_2$, whereas these are actually equal by pairs, and the notation $\Sigma x_1^2x_2^2x_3$ would be interpreted as including but one of each pair, that is, as being equal to exactly one half of $F_3(a_1^2, a^2, a)$.

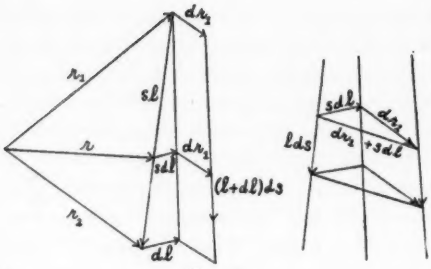


FIG. 1.

giving the differential vectors composing $\boldsymbol{dr}$. Of these $\boldsymbol{lds}$ is parallel to $\boldsymbol{l}$, and this with the vector sum of $\boldsymbol{sdl}$ and $\boldsymbol{dr_1}$ may be taken as the sides of a differential parallelogram of area proportional to the cross product of the two, so that, S being the surface generated, we have

$$dS = [\boldsymbol{lds} \times (\boldsymbol{sdl} + \boldsymbol{dr_1})]_\circ,$$

the subscript $\circ$ indicating the absolute value of the cross product which in itself is a vector. Then

$$S = \int [\boldsymbol{lds} \times (\boldsymbol{sdl} + \boldsymbol{dr_1})]_\circ = \int\int \left[\boldsymbol{l} \times \left(s\frac{\boldsymbol{dl}}{dt} + \frac{\boldsymbol{dr_1}}{dt}\right)\right]_\circ ds dt,$$

if $\boldsymbol{l}$ and $\boldsymbol{r_1}$ are functions of t. Again, since $\boldsymbol{l} = \boldsymbol{r_2} - \boldsymbol{r_1}$, we may write

$$S = \int_{t=t_1}^{t=t_2} \int_{s=0}^{s=1} \left[\boldsymbol{l} \times \left\{(1-s)\frac{\boldsymbol{dr_1}}{dt} + s\frac{\boldsymbol{dr_2}}{dt}\right\}\right]_\circ dt ds,$$

the limits 0 and 1 for s carrying the differential of surface the total length of the vector $\boldsymbol{l}$, and the limits t_1 and t_2 for t determining the first and final positions of the vector $\boldsymbol{l}$.

The beauty and simplicity of this method of attack may be best illustrated by an application. Let us apply it to the problem of finding the lateral surface of a frustum of a circular cone.

Let the radii of the bases be R and r and the height of the frustum h, the projection of the center of the upper base being at a distance a from the center of the lower base. Using mutually perpendicular unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$, parallel, respectively, to the x, y, z-axes, we have

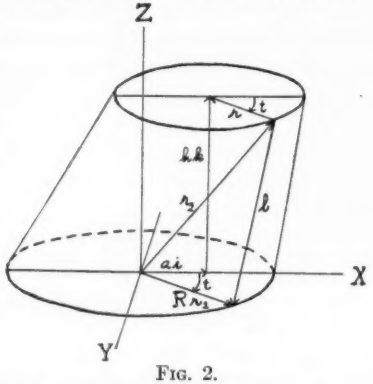


FIG. 2.

$$\boldsymbol{r_1} = \boldsymbol{i}R\cos t + \boldsymbol{j}R\sin t, \qquad \boldsymbol{r_2} = \boldsymbol{i}(a + r\cos t) + \boldsymbol{j}r\sin t + \boldsymbol{k}h,$$

t being the angle between the $x - z$ plane and the intersection of the plane of either base with the plane containing the line element l and the centers of the bases. From these,

$$\frac{\boldsymbol{dr_1}}{dt} = -\boldsymbol{i}R\sin t + \boldsymbol{j}R\cos t, \qquad \frac{\boldsymbol{dr_2}}{dt} = -\boldsymbol{i}r\sin t + \boldsymbol{j}r\cos t;$$

whence

$$(1-s)\frac{\boldsymbol{dr_1}}{dt} + s\frac{\boldsymbol{dr_2}}{dt} = \{R(1-s) + rs\}\{-\boldsymbol{i}\sin t + \boldsymbol{j}\cos t\}.$$

Now

$$\boldsymbol{l} = \boldsymbol{r}_1 - \boldsymbol{r}_2 = \boldsymbol{i}\{(R - r)\cos t - a\} + \boldsymbol{j}(R - r)\sin t - \boldsymbol{k}h$$

and, therefore,

$$\boldsymbol{l} \times \left\{ (1 - s)\frac{d\boldsymbol{r}_1}{dt} + s\frac{d\boldsymbol{r}_2}{dt} \right\}$$
$$= \{R(1 - s) + rs\}\{\boldsymbol{i}h\cos t + \boldsymbol{j}h\sin t + \boldsymbol{k}(R - r - a\cos t)\}$$

and

$$\left[\boldsymbol{l} \times \left\{ (1 - s)\frac{d\boldsymbol{r}_1}{dt} + s\frac{d\boldsymbol{r}_2}{dt} \right\} \right]_0 = \{R(1 - s) + sr\}\sqrt{[h^2 + (R - r - a\cos t)^2]};$$

whence

$$\frac{S}{2} = \int_0^{\pi}\int_0^1 \{R(1 - s) + sr\}\sqrt{[h^2 + (R - r - a\cos t)^2]}\,dt\,ds$$
$$= \frac{R + r}{2}\int_0^{\pi} \sqrt{[h^2 + (R - r - a\cos t)^2]}\,dt,$$

so that

$$S = (R + r)\int_0^{\pi} \sqrt{[h^2 + (R - r - a\cos t)^2]}\,dt.$$

This is an elliptic integral. We may write $S = (R + r)\sigma$ where σ is the length between $x = 0$ and $x = \pi h$ of the curve

$$y = \frac{R - r}{h}x - a\sin\frac{x}{h}.$$

There are a number of interesting special cases.

1. If $R = r$, $S = 2R\int_0^{\pi}\sqrt{(h^2 + a^2\cos^2 t)}\,dt = L\sqrt{(a^2 + h^2)}$, where L is the length of the elliptic cross section of axes $2R$ and $2Rh/\sqrt{(a^2 + h^2)}$ of the circular cylinder.

2. If $R = r$ and $a = 0$, $S = 2\pi RL$, the lateral area of a right circular cylinder.

3. If $h = 0$,

$$S = (R + r)\int_0^{\pi}(R - r - a\cos t)\,dt$$
$$= \pi(R^2 - r^2),$$

the difference in area of two circles of radii R and r lying in the base plane.

4. If $a = 0$,

$$S = (R + r)\int_0^{\pi}\sqrt{[h^2 + (R - r)^2]}\,dt$$
$$= \pi(R + r)\sqrt{[h^2 + (R - r)^2]},$$

the lateral surface of the frustum of a right circular cone.

5. If $r = a = 0$, $S = R\int_0^{\pi}\sqrt{(h^2 + R^2)}\,dt = \pi R\sqrt{(h^2 + R^2)}$, the lateral surface of a cone of revolution.

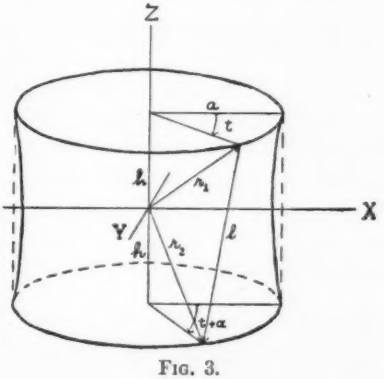


FIG. 3.

6. If $r = a = h = 0$, $S = R\int_0^\pi R dt = \pi R^2$, the area of the base.

7. If $a = 0$, and r be replaced by $-r$, $S = (R-r)\int_0^\pi \sqrt{[h^2 + (R+r)^2]}dt = \pi(R-r)\sqrt{[h^2+(R+r)^2]}$, the difference of the lateral surfaces of two cones of revolution of radii R and r and altitudes $Rh/(R+r)$ and $rh/(R+r)$.

Another interesting application, where the integration can be completed for all cases, is that of finding the surface generated by a line joining points on the circumference of the bases of a right circular cylinder, of radius a and height $2h$, differing in angular position by an angle α.

In this problem, assuming axes as shown in figure 3, we have

$$\boldsymbol{r}_1 = \boldsymbol{i}\cdot a\cos t + \boldsymbol{j}\cdot a\sin t + \boldsymbol{k}\cdot h,$$
$$\boldsymbol{r}_2 = \boldsymbol{i}\cdot a\cos(t+\alpha) + \boldsymbol{j}\cdot a\sin(t+\alpha) - \boldsymbol{k}\cdot h,$$
$$\boldsymbol{l} = \boldsymbol{r}_1 - \boldsymbol{r}_2 = 2a\left\{\boldsymbol{i}\cdot\sin\frac{\alpha}{2}\sin\left(t+\frac{\alpha}{2}\right) - \boldsymbol{j}\cdot\sin\frac{\alpha}{2}\cos\left(t+\frac{\alpha}{2}\right)\right\} + 2\boldsymbol{k}\cdot h;$$

whence

$$\boldsymbol{l}\times\left((1-s)\frac{d\boldsymbol{r}_1}{dt} + s\frac{d\boldsymbol{r}_2}{dt}\right)$$
$$= 2\boldsymbol{i}\cdot ah\left\{\cos t - 2s\frac{\alpha}{2}\sin\left(t+\frac{\alpha}{2}\right)\right\} + 2\boldsymbol{j}\cdot ah\left\{\sin t + 2s\sin\frac{\alpha}{2}\cos\left(t+\frac{\alpha}{2}\right)\right\}$$
$$+ 2\boldsymbol{k}\cdot(1-2s)\sin^2\frac{\alpha}{2},$$

giving

$$S = \int_0^{2\pi}\int_0^1 2a\sqrt{\left[h^2\cos^2\frac{\alpha}{2} + (2s-1)^2\sin^2\frac{\alpha}{2}\left(h^2 + a^2\sin^2\frac{\alpha}{2}\right)\right]}dtds$$
$$= 2\pi a\sqrt{\left(h^2 + a^2\sin^2\frac{\alpha}{2}\right)}$$
$$+ \frac{2\pi ah^2\cos^2\frac{\alpha}{2}}{\sin\frac{\alpha}{2}\sqrt{\left(h^2+a^2\sin^2\frac{\alpha}{2}\right)}}\log\left\{\frac{\sin\frac{\alpha}{2}\sqrt{\left(h^2+a^2\sin^2\frac{\alpha}{2}\right)} + \sqrt{\left(h^2+a^2\sin^4\frac{\alpha}{2}\right)}}{h\cos\frac{\alpha}{2}}\right\}.$$

If $\alpha = \pi$ the surface becomes that of two cones, or $S = 2\pi a\sqrt{(a^2+h^2)}$.

If $\alpha = 0$ the surface is that of the cylinder of revolution. The limit of the second term for $\alpha \doteq 0$ is $2\pi ah^2(1/h)$. Hence we get $S = 4\pi ah$.

Again if $h = a\sin\alpha/2$,

$$S = \pi a^2\left\{2\sin\frac{\alpha}{2}\sqrt{\left(1+\sin^2\frac{\alpha}{2}\right)} + \sqrt{2}\cos^2\frac{\alpha}{2}\log\frac{\sqrt{\left(1+\sin^2\frac{\alpha}{2}\right)} + \sqrt{2}\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}\right\},$$

which is the surface of the hyperboloid of one sheet generated by revolving

the hyperbola $x^2 - y^2 = a^2 \cos^2 \frac{1}{2}\alpha$ about the y-axis from $y = -a \sin \alpha/2$ to $y = +a \sin \alpha/2$.

This method may be followed in finding a definite portion of a tangent surface of a space curve or in finding the surface generated by a given part of the normal or binormal to a curve.

For a tangent surface, since $\boldsymbol{l}$ is a portion of the line tangent to the curve defining $\boldsymbol{r}_1$, $\boldsymbol{l}$ is parallel to $\boldsymbol{dr}_1/dt$ so that $\boldsymbol{l} \times \boldsymbol{dr}_1/dt = 0$, simplifying the formula to

$$S = \int_{t_1}^{t_2} \int_0^1 \left(\boldsymbol{l} \times \frac{\boldsymbol{dl}}{dt} \right)_0 s dt ds = \tfrac{1}{2} \int_{t_1}^{t_2} \left(\boldsymbol{l} \times \frac{\boldsymbol{dl}}{dt} \right)_0 dt.$$

For example, to find the area of the surface generated by the length a of the tangent to the helix $\boldsymbol{r} = \boldsymbol{i} \cdot a \cos t + \boldsymbol{j} \cdot a \sin t + \boldsymbol{k} \cdot ct$ in one revolution, we have

$$\frac{\boldsymbol{dr}}{dt} = -\boldsymbol{i}a \sin t + \boldsymbol{j}a \cos t + \boldsymbol{k}c;$$

whence

$$\boldsymbol{l} = \frac{a}{\sqrt{(a^2 + c^2)}} (-\boldsymbol{i}a \sin t + \boldsymbol{j}a \cos t + \boldsymbol{k}c),$$

the absolute length being a, and, therefore,

$$\frac{\boldsymbol{dl}}{dt} = \frac{a}{\sqrt{(a^2 + c^2)}} (-\boldsymbol{i}a \cos t - \boldsymbol{j}a \sin t)$$

and

$$S = \frac{1}{2} \frac{a^3}{a^2 + c^2} \int_0^{2\pi} [-\boldsymbol{i}c \sin t - \boldsymbol{j}c \cos t + \boldsymbol{k}a]_0 dt$$

$$= \frac{1}{2} \frac{a^3}{a^2 + c^2} \int_0^{2\pi} \sqrt{(a^2 + c^2)} dt = \frac{\pi a^3}{\sqrt{(a^2 + c^2)}}.$$

In case $c = 0$ we get $S = \pi a^2$, the area of the annulus of outer radius $\sqrt{2}a$ and inner radius a.

AN ENGLISH TEXT ON MATHEMATICS WRITTEN ABOUT 1810.[1]

By ELIZABETH B. COWLEY, Vassar College.

This three-volume treatise is owned by Dr. Charles C. Godfrey of Bridgeport, Conn., who purchased the books in a second-hand store in New Haven. He says that they were part of a large library of works on mathematics collected by an Englishman and sold as part of his estate and later brought to this country. He has recently written that he has learned that these three volumes once belonged to Mr. Thomas P. Stowell of Rochester, N. Y., who was probably connected with the Alexandria Boarding School.

These three bound volumes are written by hand in ink. Many of the problems

[1] Read at the Summer meeting of the Association, Rochester, N. Y., Sept. 6, 1922.

have elaborately drawn illustrations colored in water colors; and each volume has a full page fanciful frontispiece in colors. The pages are not numbered consecutively in ink, but on the first page "No. 1" is written in pencil in a large hand; on the seventeenth page "No. 2"; on the thirty-third page "No. 3," etc. This pencil numbering is found in each volume, each number usually indicating 16 pages. The last number in vol. 1 is 38; in vol. 2 is 29; and in vol. 3 is 30.

There is no statement of date, place, or author. I have not found any printed text of which this is a copy, although some parts of it bear a resemblance to portions of the *Mathematical Institutions* of W. Leybourn, which was published in London in 1704. In volume 3 there is written in a small hand in ink in the upper right-hand corner of the first fly-leaf

W. B. Falcon
May 20th, 1811
Whitehaven.

This date and place fit in well with some stray bits of internal evidence. For example, one problem tells of Allonby-bay on the coast of Cumberland and of houses across on the Scotch shore. A rebus refers to Carlisle. The description of the "Carpenter's Rule, commonly called Coggeshall's sliding rule" agrees with the description found in Barlow's *Dictionary* (1814). The illustrated problem on the coal wagon would not be out of place in that mining district. Volume 3, which starts out with "Book keeping Moderniz'd," begins the "Waste Book" the first of January 1807, London. Volume 2 gives the "Journal of a Voyage from London to Madeira and Teneriffe in the Heroine of Workington, Thomas Falcon Commander," which begins April 12, 1807. There may or may not be any significance in these two references to London.

Volume 1 begins with "Practical Geometry." From the 42 definitions I quote two: "A right line is that which lies evenly between its extreme points." "The measure of an angle is the arc of a circle contained between the two lines which form that angle, the angular point being the centre." There are 51 problems. The first 40 of these are ordinary constructions of plane geometry. The next 10 are problems in which, when the measurements of certain sides or angles of right or oblique triangles are given, it is required to construct the triangle and measure the other parts. The 51st problem is "To make a plane scale or to construct a scale of Sines, Tangents, Chords, Semitangents, Longitudes, and Rhumbs to any radius."

This "Practical Geometry" serves as an introduction to "Plain Trigonometry," which starts in with definitions of trigonometry and of "the lines applied to a circle." For example, "the right sine of an arc is a right line drawn from one end or termination of an arc to the radius and perpendicular thereto." There are 6 cases of right-angled triangles, each of which is solved by 3 methods:—by geometrical construction, by Gunther's scale and compasses, and by logarithmic sines, tangents, etc. There are 4 precepts which the writer says "must be well impressed upon the memory of all who wish to be ready and dexterous in this most excellent art": 1. "When a side is required, begin the operation with an

angle and when an angle is sought begin with a side." 2. "As the sine of an angle : its opposite side :: the sine of any other angle : its opposite side." 3. "As either leg : tangent of the radius 45° :: the other leg : tangent of the other opposite angle." 4. "Extend the first term to the third and that extent will reach from the second to the fourth." The four cases of the oblique triangle may be solved by these and two new precepts. The first is our ordinary law of tangents, stated as a proportion. The second precept, which may be reduced to our law of cosines, is stated thus : "As the longest side : sum of the other two sides :: the difference of the said two sides : the difference of the segments of the base made by a perpendicular let fall from the vertical angle."

"Altimetry and Longimetry" deal with the applications of trigonometry. There are 18 problems, each illustrated by an elaborate and carefully drawn ink picture, colored with water colors. Half of the problems deal with the ever-popular theme of towers. Their pictures are varied—including round and square, ruined and well-preserved, some set on hills and others beside the sea or a stream. From one tower the British flag is flying. We find also a May pole, a church steeple, a number of fully rigged sailing vessels, and houses. In one case the houses on both sides of a street are drawn. The problem is the familiar one dealing with the ladder that leans first against a house on one side of the street and then is turned over to lean against one on the other side. In addition to these 18 problems collected here, there are others amongst the miscellaneous problems at the end of volume 2. These involve castles, ships, and farm houses. All these problems would repay study, but we shall mention here only the twelfth in the first collection. It is the most elaborate and it has so much local color. On the seacoast of Cumberland stand Dubmill and Bankend on opposite sides of Allonby-bay from each of which appear two houses, which for distinction we call A and B, on the Scotch shore, the distances of which from Dubmill and Bankend and from one another are required from the data subjoined. Suppose at Dubmill the line to house B bisects the angle there between Bankend and house A, then admit 80 chains measured from Dubmill on the line from house A so that a line to Bankend was perpendicular thereto, and there a signal put up or a fire made. Next at Bankend suppose the following angles observed:

1. Between the signal or fire and Dubmill 12° 30′.
2. Between Dubmill and house A 54° 30′.
3. Between houses A and B 61° 30′.

The water color illustration is elaborate, showing Dubmill with its wheel, and the other places mentioned.

The next subject taken up is the "Mensuration of Superficies." This involves areas of parallelograms, triangles, and other polygons, of circles, sectors and segments of circles, and the Pythagorean theorem. No attempt is made to prove a theorem, instead there are "rules."

"Conic sections" is devoted to definitions and the problems of constructing and finding the areas of segments of the ellipse, parabola, and hyperbola. The mensuration of solids deals with the convex surface and solidity of the parallel-

opipedon, cone, sphere, spheroid, parabola and hyperbola of revolution, parabolic spindle and the "cylindrical ring." In each case there is a rule followed by examples solved by the rule.

Artificers' work includes that of bricklayers, masons, carpenters and joiners, slaters and tilers, plasterers, painters and glaziers, etc. There is a description of the carpenters' rule with examples on timber measure, etc. The last half of the first volume is devoted to spherics, including stereographic projection, spherical trigonometry, and astronomical problems solved by the globe construction and by calculations.

Volume 2 treats of navigation, starting in with sailing:—plane, traverse, parallel, middle latitude, etc. There are other problems such as "Construction and use of maps." It is in this volume that we find the journal of the voyage of the "Heroine of Workington," which I have previously mentioned. At the end of the volume there is an odd assortment of questions which are solved by algebra and there are several rebuses. One problem and one rebus refer to "Green Row." I shall be glad to have any suggestion as to what "Green Row" is.

Volume 3 opens with "Bookkeeping Moderniz'd." There are a "Waste Book," Journal, Index, and Ledger. This volume also contains algebra, with questions involving "simple" and quadratic equations. It is concluded by a bit on converging series and some "promiscuous questions" and applications of algebra to geometry. The first part of the book is the most interesting. The "Waste Book" is dated London, the first of January 1807. It starts out "Inventory of money, goods and debts belonging to me *AB*, as also of the debts due by me to others." As we read we learn that "*AB*" is a merchant doing business with Liverpool, Bristol, Glasgow, Edinburgh, Dublin, Cork, etc., as well as with foreign ports such as New York, Boston, Philadelphia, Amsterdam, Funchall, Cadiz, Hamburg, Leghorn, Smyrna, etc. He deals in many different kinds of merchandise extending from "blue Hungary pearl ashes," broadcloth, fustians, kerseys, linen, serges, figs, flour, flax, hops, Canary wine, rum, tobacco, sugar, raisins, herring, beef, pork, and shoes. He had a house in Fleet St. and another in Charing Cross and he paid house rent and window tax. He purchased two lottery tickets and he gained a wager on the safe arrival in Smyrna of the ship "Happy Janet." He was interested in Drummond's shoe factory and in the Royal Bank of Edinburgh: he opened a store in Boston and he had part interest in several ships. On May 9 he enters the statement "George Barclay is broke." Even the list of family names of those with whom he transacted business is interesting. The last entry for Dec. 31 is "Laid out for the use of my family since the first of January last £300."

In conclusion may I suggest that the usefulness of old books like these is not confined to the person who has a special interest in the history of mathematics. Last year I showed these books to my freshmen when they were studying trigonometry. They were especially interested in the problems on heights and distances and were eager to solve them and compare their solutions with those given. Later I showed them the photostats of a fourteenth-century manuscript

from Italy upon which I was working. From glimpses of old books and manuscripts they begin to realize that mathematics is not a cold, dead subject the same to-day as it was 1000 years ago and as it will be centuries hence; but rather that it is closely interwoven with the life of any period and that there is a mutual dependence between the progress of mathematics and that of civilization.

A GEOMETRIC PARADOX.

By B. H. BROWN, Dartmouth College.

We are indebted to Professor Coolidge for the remark that "there is nothing more instructive than an apparent paradox." The following situation involves interesting theorems, the answer is not too obvious, and the student may find in the simple explanation a concept which is new to him.

One of the curious facts in elementary geometry is that a plane bitangent to a torus cuts it in two circles, a theorem due to Villarceau.[1] Hence, in addition to the meridian and parallel circles there are two other systems of circles such that each circle of one system cuts every circle of the other system, but no circle of one system cuts any other circle of the same system. Furthermore Schoelcher[2] proved the "joli théorème" that these Villarceau circles are loxodromes of the meridian and of the parallel circles. Now consider the following chain of theorems:

1. *A cone of revolution can be inverted into a torus;*
2. *The lines of curvature on a cone of revolution are the rulings and parallel circles;*
3. *The lines of curvature on a torus are the meridian and parallel circles;*
4. *Inversion carries lines of curvature into lines of curvature;*
5. *Inversion carries circles into circles (straight lines being considered circles);*
6. *Inversion preserves angles (except for sign).*

If then we invert a cone into a torus, where can we find any circles on the cone to invert into the Villarceau circles? And these circles on the cone must be loxodromes of the rulings, an exceedingly difficult thing to visualize.

Let us carry through the details of a particular transformation. The inversion with center at $(0, 0, i)$ and power -2, whose equations are

$$\bar{x} = \frac{-2x}{x^2+y^2+(z-i)^2}, \qquad \bar{y} = \frac{-2y}{x^2+y^2+(z-i)^2},$$
$$\bar{z} = i + \frac{-2(z-i)}{x^2+y^2+(z-i)^2} = \frac{i(x^2+y^2+z^2+1)}{x^2+y^2+(z-i)^2}, \tag{1}$$

will carry the (imaginary) cone

$$\bar{x}^2 + \bar{y}^2 + \tfrac{1}{2}\bar{z}^2 = 0 \tag{2}$$

into the real torus

$$(x^2+y^2+z^2+1)^2 = 8(x^2+y^2). \tag{3}$$

[1] Cf. *Nouvelles Annales de Mathématiques*, series 1, vol. 7, 1848, pp. 345–347. An elementary proof for a special case is given later in this paper.

[2] Cf. *Nouvelle s Annales de Mathématiques*, 4th series, vol. 3, 1903, pp. 105–107.

The torus (3) is formed by the revolution about the z-axis of a circle of radius unity, the plane of the circle passing through this axis, and the center of the circle lying in the xy plane at a distance $\sqrt{2}$ from the origin. The plane $x = z$ is bitangent to the torus. If we rotate the torus about the y-axis until this plane becomes the xy plane, the curve of intersection of plane and torus may be found by transforming (3) by

$$x = \bar{x}\tfrac{1}{2}\sqrt{2} - \bar{z}\tfrac{1}{2}\sqrt{2}, \qquad y = \bar{y}, \qquad z = \bar{x}\tfrac{1}{2}\sqrt{2} + \bar{z}\tfrac{1}{2}\sqrt{2},$$

and setting $\bar{z} = 0$. This gives a factorable quartic

$$(\bar{x}^2 + \bar{y}^2 - 1 + 2\bar{y})(\bar{x}^2 + \bar{y}^2 - 1 - 2\bar{y}) = 0;$$

that is, two Villarceau circles of different systems with centers on the y-axis at $(0, 1, 0)$ and $(0, -1, 0)$ and with radii $\sqrt{2}$. These two circles cut orthogonally, and by Schoelcher's theorem and a corollary thereto, the Villarceau circles of this torus cut every line of curvature under the angle 45°, and each Villarceau circle of one system cuts every circle of the other system orthogonally.

Under the inversion (1) it is clear that a plane through the z-axis is invariant and consequently the two rulings in which it cuts the cone invert into the two meridian circles in which the plane cuts the torus. Again, spheres orthogonal to the z-axis invert, in general, into spheres orthogonal to the z-axis. Such spheres cut the cone in two parallel circles, and their transforms cut the torus in two parallel circles. We leave to the reader the explanation of the following situation. A plane $z = c$, cutting the cone in one circle, inverts into a sphere normal to the z-axis cutting the torus in two circles!

But where do the Villarceau circles come from? Let us invert the torus back into the cone and see where they go. Two co-planar Villarceau circles of different systems are given by the intersection of a plane

$$x \cos\theta + y \sin\theta - z = 0, \tag{4}$$

and the torus (3). The inverse of these two circles is given by the intersection of

$$x^2 + y^2 + \tfrac{1}{2}z^2 = 0, \quad \text{and} \quad x^2 + y^2 + z^2 + 1 - 2ix\cos\theta - 2iy\sin\theta = 0,$$

or, more simply, by the intersection of

$$x^2 + y^2 + \tfrac{1}{2}z^2 = 0, \quad \text{and} \quad x^2 + y^2 + 2ix\cos\theta + 2iy\sin\theta - 1 = 0. \tag{5}$$

Now (5) is factorable into

$$(x + iy + ie^{i\theta})(x - iy + ie^{-i\theta}) = 0; \tag{6}$$

so that the inverses of one system of Villarceau circles are given as intersections of the cone and the planes

$$x + iy = c; \qquad c \neq 0, \tag{7}$$

and of the other system by the cone and the planes

$$x - iy = c; \qquad c \neq 0, \tag{8}$$

the relation between c and θ being evident from (6).

These are the transforms of the Villarceau circles, and the following reasoning

will make clear that these are circles. The intersection of the plane at infinity and the cone (2) is a conic bitangent to the Absolute at $(1, i, 0, 0)$ and $(i, 1, 0, 0)$ and the planes (7) and (8) are tangent to the Absolute at these two points respectively. The intersection of a plane (7) or (8) with the cone (2) is a most curious conic: a parabola because it is tangent to the line at infinity in its plane, but a circle because it goes through the two (coincident) circular points in its plane.

We may now verify analytically that these parabolic circles cut each other orthogonally, and that they cut the rulings and parallel circles under an angle of 45°. In fact it is easy to show that the parabolic circles on any cone of revolution

$$x^2 + y^2 + kz^2 = 0$$

cut the parallel circles under the angle $\cos^{-1}\sqrt{k}$, and that one system of parabolic circles cuts the other system under the angle

$$\cos^{-1}(1 - 2k).$$

Recalling the theorem [1] that every Dupin cyclide can be inverted into a cone of revolution we see that from the standpoint of inversion there is a single infinity of distinct Dupin cyclides characterized by the angle between their Villarceau circles.

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

DISCUSSIONS.

I. AN APPROXIMATE CONSTRUCTION OF THE SIDE OF A REGULAR INSCRIBED HEPTAGON.[2]

By T. R. RUNNING, University of Michigan.

The construction of Figure 1 gives a remarkably close approximation to the side of a regular inscribed heptagon. If the segment AP be stepped around the circumference of the circle from A seven times, the terminal point will fall short of A by an arc approximately equal to 0".000432.

If the construction be carried out for a circle the size of the earth's equator and AP stepped around the circumference seven times, the terminal point will fall short of the initial point by about 0.53 inch.

[1] Mannheim gave a very neat proof of an equivalent theorem: every Dupin cyclide can be inverted into a torus. Cf. GOURSAT-HEDRICK, *A Course in Mathematical Analysis*, vol. 1, p. 525.

[2] A number of historical references for this problem were given by R. C. Archibald in this MONTHLY (*1921*, 473-9). In particular, allusion is there made to Röber's construction, the accuracy of which (the total error being half a second) astonished Hamilton and De Morgan. Professor Running's very simple construction involves an error less than one thousandth of this.

A word may be said with regard to the idea that the simplicity of such a construction depends on the number of square roots involved (see *loc. cit.*, p. 479). This, of course, is not the case. If $\cos(2\pi/7) = m/n$ to 20 decimal places, the first degree equation $n\cos(2\pi/7) = m$ suffices within these limits without the taking of any square roots. In the present construction, the steps are both few and simple, the principal labor being the division of two segments each into seven equal parts. EDITOR.

Figure 1 shows the construction. The point F, determined as indicated is joined to E by a straight line. The diameters, AE and LM, are perpendicular to each other. The point S is the intersection of FE with the line HS parallel to LM. The line SC continued intersects AK in D. CD is bisected in Q, and the line AQ drawn, intersecting OB in P. AP is the approximate side of a regular heptagon inscribed in the circle whose radius is AO.

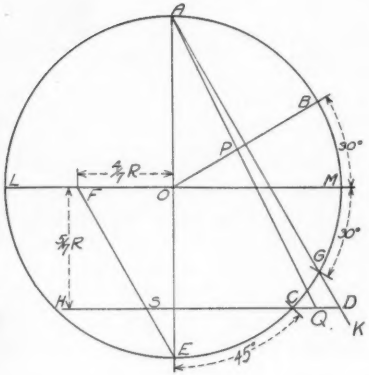


FIG. 1.

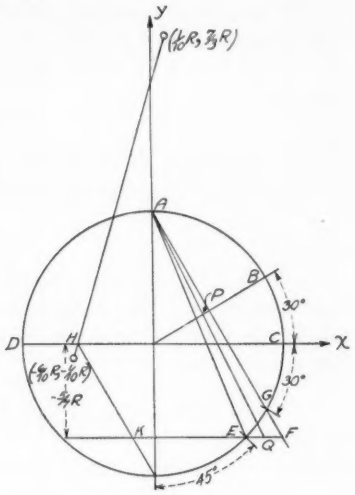


FIG. 2.

It is to be noted that HS and SC do not form the same straight line.

A modified construction which differs from this only in the addition of one more line is indicated in Figure 2.

The point H is determined as the intersection of the line joining $(\frac{1}{10}R, \frac{7}{5}R)$ and $(-\frac{6}{10}R, -\frac{1}{10}R)$ with DC. The rest of the construction is the same.

In this case the defect is only 0".000185 or 0.23 inch.

II. Evaluation of the Determinant $|1/(r+s-1)!|$.

By L. L. Dines, University of Saskatchewan.

I have recently had occasion to evaluate the determinant

$$D_n = \begin{vmatrix} \frac{1}{1!} & \frac{1}{2!} & \cdots & \frac{1}{n!} \\ \frac{1}{2!} & \frac{1}{3!} & \cdots & \frac{1}{(n+1)!} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \frac{1}{n!} & \frac{1}{(n+1)!} & \cdots & \frac{1}{(2n-1)!} \end{vmatrix}.$$

Although it belongs to the class of determinants sometimes called *orthosymmetrical*, and specially studied by Hankel,[1] the principal theorem relating to this class of determinants is not directly effective. Inasmuch as I have not found the evaluation of D_n nor of any determinant including it as a special case [2] in the books I have at hand, it may be of some interest to record it.

If the elements of each column of D_n be multiplied by the reciprocal of the last element in that column, and compensation be made by means of appropriate multipliers for the determinant, we may write

$$D_n = \frac{1}{n!(n+1)!\cdots(2n-1)!}\,|a_{rs}|_{(r,s=1,2,\cdots,n)},$$

where

$$a_{rs} = \frac{(n+s-1)!}{(r+s-1)!}.$$

Every element of the last row of the determinant $|a_{rs}|$ is equal to 1. Hence it may be reduced to a determinant in which all elements of the last row are zero with the exception of the last one, by subtracting the second column from the first, the third from the second, and in general the $(s+1)$th column from the sth column for $s = 1, 2, \cdots, n-1$. By this transformation, the element in the rth row and sth column becomes

$$\frac{(n+s-1)!}{(r+s-1)!} - \frac{(n+s)!}{(r+s)!} = -(n-r)\frac{(n+s-1)!}{(r+s)!} \quad (r, s = 1, 2, \cdots, n-1);$$

and by taking the factor $-(n-r)$ out of the rth row for $r = 1, 2, \cdots, n-1$, we may reduce the determinant $|a_{rs}|$ as follows:

$$|a_{rs}|_{(r,s=1,2,\cdots,n)} = (-1)^{n-1}(n-1)!\,|a_{rs}^{(1)}|_{(r,s=1,2,\cdots,n-1)}$$

where

$$a_{rs}^{(1)} = \frac{(n+s-1)!}{(r+s)!}.$$

Each element of the last row of $|a_{rs}^{(1)}|$ is equal to 1. Hence a transformation similar to that used on $|a_{rs}|$ will suffice to reduce its order by one. We find

$$|a_{rs}^{(1)}|_{(r,s=1,2,\cdots,n-1)} = (-1)^{n-2}(n-2)!\,|a_{rs}^{(2)}|_{(r,s=1,2,\cdots,n-2)}$$

where

$$a_{rs}^{(2)} = \frac{(n+s-1)!}{(r+s+1)!}.$$

Again, the elements of the last row of $|a_{rs}^{(2)}|$ are equal to 1. The process of reduction of order can be repeated successively, the jth repetition giving the

[1] See Kowalewski's *Einführung in die Determinantentheorie*, page 112.

[2] For the evaluation of determinants of a very similar character, see however Scott's *Treatise on the Theory of Determinants*, page 79.

reduction formula

$$|a_{rs}^{(j-1)}|_{(r,s=1,2,\cdots,n-j+1)} = (-1)^{n-j}(n-j)!\,|a_{rs}^{(j)}|_{(r,s=1,2,\cdots,n-j)}$$

where

$$a_{rs}^{(j)} = \frac{(n+s-1)!}{(r+s+j-1)!}.$$

After $n-1$ reductions we obtain a determinant of a single element

$$(-1)\,|a_{rs}^{(n-1)}|_{(r,s=1)} = -1.$$

Assembling the results of the successive reductions, we have [1]

$$D_n = (-1)^{\frac{n(n-1)}{2}} \frac{1!\,2!\,3!\cdots(n-1)!}{n!(n+1)!\cdots(2n-1)!}.$$

III. A Correction.

By R. M. Mathews, Wesleyan University.

Dr. B. H. Brown has called my attention to an error caused by uncritical zeal in my note in the October (1922) Monthly.[2] The following paragraph on p. 348 should be deleted:

"It is evident that the main theorem carries on to curves of still higher degree obtained by successive inversions."

The statement is true but vacuously so, for an inversion does not raise the order of a bicircular quartic; the curve already passes twice through the circular points at infinity, and a change of the pole of inversion does not change these points. In the general case of a quadratic transformation (of which inversion is a special case) successive transformations will double the order of the curve each time if we take a fundamental triangle entirely different from the preceding.

The error in question is found in the article by Cazamian as cited.

RECENT PUBLICATIONS.

REVIEWS.

Einleitung in die Mengenlehre. By A. Fraenkel. Berlin, Julius Springer, 1919. vi + 156 pp.

This book is an introduction to the researches of G. Cantor—a field which, even among professional mathematicians, has a special reputation for subtleties and abstractions. And yet, with all of its dignity of subject-matter and of style,

[1] A somewhat different treatment would be to multiply the rth row by $(n+r-1)!$, and divide the sth column by $(n-s)!$, for every r and s. The elements are then binomial coefficients, and the determinant as modified easily found to be ± 1. This same method evaluates any minor of D_n standing in the upper right-hand corner. Several references to determinants involving binomial coefficients may be found in E. Pascal, *I Determinanti*, Milan, 1897, pp. 173, 175. Editor.

[2] "Concyclic Points on an Equilateral Hyperbola and its Inverses," this Monthly (*1922*, 347–8).

Fraenkel's *Einleitung* is intended for the mathematical amateur. It is thus a contribution to the scanty field of popular literature that is concerned with new mathematical discoveries. Nearly every person with a capacity for abstract thinking can read it with understanding. The actual mathematical knowledge needed by the reader barely goes beyond high-school algebra. For example, a proof is inserted of the irrationality of $\sqrt{2}$; and more than a page of fine type is devoted to showing that there exist only a finite number of algebraic equations, with integral coefficients, having a given rank (*Höhe*)—the rank of the polynomial $a_0x^n + a_1x^{n-1} + \cdots + a_n$ being defined as $(n-1) + |a_0| + |a_1| + \cdots + |a_n|$. Of course, it is to be expected, since the book is written in German, that the popularity is of the German kind. And so it is: there is no soft-pedalling of abstractness or of Gründlichkeit.

The style is clear and vivid; concrete examples are copious, especially in the earlier portion. The sentences are wrought with a literary and logical swing that must have given joy to the author. There is a freedom and a warmth that may be ascribed in part, no doubt, to the way in which the book originated, "the incentive came in conversations with comrades of war (non-mathematicians), whose desolate hours I could now and then abbreviate by initiating them in the thought-paths of the theory of aggregates." The author succeeds well in conveying the beauty, the sweep and the revolutionary significance of Cantor's creation.

The reasons why Cantor's work has an uncommon attraction for the mathematical amateur are apparent. It is the first rigorous study of the infinite as such; it is singularly free from technical matter and formulas; it is the groundwork of other mathematical disciplines—scarcely any branch of mathematics has been left unaffected by its ideas; it is the closest to logic and to philosophy. Then, too, it is only through an acquaintance with the theory of aggregates that one can appreciate the nature of the modern controversies concerning the infinite, which have divided mathematicians into two or more opposing camps.

The following topics are discussed: aggregate (naïve definition), equivalence,[1] denumerable sets,[2] cardinal number of the continuum and of the set of real functions, calculation with infinite cardinals, ordinal types and ordinal numbers, linear sets, especially with reference to the order types η and λ, and normal order. There follows a consideration of certain paradoxes in logic and in the theory of aggregates, a critique of the concept of set and of the Auswahlprinzip, and a sketch of Zermelo's *Grundlagen*.

On the question of Wohlordnung, Fraenkel takes a decided view—in contradistinction to the doubts in Hobson's recent volume: "There seems to be no good reason for regarding the Auswahlprinzip as less evident than every other axiom

[1] Two sets M and N are said to be equivalent if a correspondence can be set up between them in such a way that to every element of M there corresponds just one element of N, and to every element of N, just one element of M. The mating of the positive integer n with the positive even integer $2n$ shows that if M is the set of positive integers, and N the set of positive even integers, then M and N are equivalent.

[2] Sets equivalent with the set of positive integers.

of mathematics." And as to rigor in the theory of aggregates, he has this to say: "We thus have good ground for regarding the Cantor-Zermelo theory of aggregates as a discipline possessing the same rigor and security as every other branch of mathematics; and the transfinite cardinals and ordinals as possessing in the mathematical-logical kingdom, just as any other mathematical-logical concept, the undiminished rights of citizenship."

The author's interest in logical refinement results in greater attention to abstract questions than may be justified on the ground of promoting the reader's understanding. For example, the space spent in showing in detail (p. 11) that if a set M is equivalent to a set N, then N is equivalent to M, might easily have been used for a more stimulating discussion.

The proof of the non-denumerability of the continuum[1] may be made a little more convincing to the general reader if the *reductio ad absurdum* is avoided and the following statement adopted: For every given infinite sequence S of real numbers, there is a real number not contained in S. (The same remark applies to the proof of the equivalence of the set of real numbers and the set of real functions.) The proof would also be a little simpler if the digits 1 and 2 were used for $\alpha_1, \beta_2, \gamma_3, \cdots$ (p. 34) instead of all the possible digits; subsequent explanations (p. 35) would thus be rendered largely unnecessary. There would be, however, a slight loss in naturalness.

In connection with the definition of cardinal number, it is safer to avoid telling just what a cardinal number is and to state merely when two sets "have the same cardinal number." The author's definition (p. 41) of cardinal number as the common characteristic of equivalent sets is obviously objectionable.

On p. 135 occurs the statement: "We choose as axioms propositions that are as evident as possible." If properly understood, this statement is acceptable; but if the reader is not told also of the definitional character of an axiom, he may not see that an axiom as such is neither true nor false. On p. 136 occurs the statement: "Then each of these three axioms is free from contradiction; nevertheless, when taken together, they contain a contradiction." This is not true. The axioms are consistent but they are not simultaneously valid in the case of Euclidean geometry.

There is one other point in connection with foundations that deserves mention. A number of mathematicians, following Hilbert,[2] have taken the view that the so-called *axiomatic method*—as contrasted with the so-called *genetic method*—is the method *par excellence* for the attainment of rigor.[3] This view is, of course, not justified. The proof of the Wohlordnungssatz, for example, does not gain in convincingness simply because an axiom—the multiplicative axiom—is formu-

[1] This asserts that no sequence $a_1, a_2, a_3, \cdots$ exists such that every real number equals at least one a_n.

[2] Cf. Hilbert, *Grundlagen der Geometrie*, third edition (1909), p. 256.

[3] In the case of the foundations for the system of real numbers, the genetic method would start with the positive integers and successively define the negative integers, the rational numbers, and the irrational numbers. The axiomatic method would start with an aggregate S and two undefined operations $+$ and $\times$, and would postulate enough about $(S, +, \times)$ to prove all other known facts concerning the continuum of real numbers.

lated; our conviction depends rather on the way in which this axiom appeals to us intuitively. The *Grundlagen* of Zermelo constitute a more rigorous basis for the theory of aggregates than the treatment of Cantor, not simply because they are formulated as axioms, but mainly because they restrict the definition of the term aggregate. In this matter, it would be hard for the reader to get the right point of view from Fraenkel's book.

There is one further question the reviewer wishes to raise. To obtain clearness of exposition, the author has adopted the method of long explanations. Does clearness, however, necessitate long explanations? Perhaps so, in an oral discourse; but a reader has more time to think than a listener. Here and there, in the reviewer's opinion, Fraenkel might have conserved the reader's energy by being more brief; and the space thus gained might have been utilized for applications.

HENRY BLUMBERG.

La Composition de Mathématique dans l'examen d'admission a l'École Polytechnique, 1901–1921. By F. MICHEL and M. POTRON. Paris, Gauthier-Villars. 1922. 452 pages. Price 40 francs.

As the title indicates, this book is primarily a collection of the mathematical problems set at the examinations for admission to l'École Polytechnique, 1901–1921. Complete solutions are given, many of them with bibliography. A second part of the book is a collection of exercises proposed by the authors. These are chosen with the purpose of illustrating the principles of the problems of the examinations. References to the examination problems on which they bear are given.

As to the examinations themselves: they seem to be in no way questions about mathematical facts nor do they call for proofs of well-known theorems. Problems requiring genuine ability are set. These are chosen from fields that are varied but are, nevertheless, such as should be familiar to a good first-year graduate student in an American university. The majority are of graduate difficulty although some easier ones occur.

The book, apart from its avowed purpose of aiding candidates in their preparation for a particular examination, should be of interest to Americans as a study in French education and the problems as suggesting, through analogy and generalization, comparatively simple fields for investigation.

TOMLINSON FORT.

A Course in Analytic Geometry. By P. P. BOYD, J. M. DAVIS and E. L. REES. New York, D. Van Nostrand Co., 1922. Price $2.40.

The authors, who are all professors in the University of Kentucky, state in their preface:

"The arrangement of the material of analytics is usually somewhat artificial and not in accordance with any underlying idea or principle. In this course the two fundamental problems, the locus of an equation and the equation of a locus, are given their due prominence and made the basis for arrangement of the work in both plane and solid analytics. This conduces to clarity,

makes possible effective instruction in the methods of attack peculiar to analytics and affords the advantage of a repeated study of conic sections and other important curves.

"Attention is called, among other things, to the following: The prominence given the function concept—curves are classified according to the character of the function involved rather than by the degree of the equation—the careful instruction in the discussion of an equation and in sketching, the use of the derivative for turning points and tangents, the simplicity of the conventions concerning angles, continual employment of both rectangular and polar coördinates, the large number of examples and problems and the arrangement of the material of solid analytics on the same plan as that employed in plane analytics."

The work is divided as follows: Plane analytic geometry—I. Introduction, 35 pages; II. Locus of an equation, 78 pages; III. Equation of a locus, 52 pages; IV. Analytic treatment of geometric properties, 26 pages; Solid Analytic Geometry —I. Introduction, 10 pages; II. Locus of an equation, 22 pages; III. Equation of a locus, 22 pages.

In the introduction, the authors follow the usual plan of presentation of coördinate systems and their interrelation, length of a line-segment in rectangular, oblique and polar coördinates, point of division, slope, and angle between two lines. In this last, the usual form is given for $\tan \varphi = \tan (\theta_1 - \theta_2)$ but it is specified that θ_1 is always greater than θ_2, thus making φ always positive. This is done "to avoid confusion" but the restriction is rather artificial and, in the opinion of the writer, unnecessary.

As is evident from the brief outline given above, much more space is devoted to the discussion of the locus of an equation than in most texts. The points in the discussion are: intercepts, vertical asymptotes, symmetry, limits, behavior at infinity, and later, turning points. Asymptotes are defined on page 41 with particular reference only to vertical asymptotes, yet, in Figs. 33 and 34 on page 42, horizontal asymptotes are indicated. Also later, page 79, one finds the usual reference to asymptotes in connection with the hyperbola and is lead to wonder if the student may not confuse this idea with that suggested by vertical asymptotes. Might it not be better to define and illustrate asymptotes in general and then pass to the special cases of vertical and horizontal?

In connection with the application of the derivative of a function to find the turning points, the student is asked, page 54, to show that a curve may have a horizontal tangent without having a turning point. Later, page 69, in the discussion of multiple roots, one reads: "That the curve is tangent to the x-axis at points corresponding to multiple roots (both an even and an odd number of equal roots) can be shown by proving that at these points the derivative is equal to zero." This introduces to the student a new idea of tangency and may lead to confusion. Since higher derivatives are necessary to study fully the matter of maxima, minima and points of inflection, the writer feels that this whole question, as far as analytic geometry is concerned, should be left with merely the statement that the values of x for which the first derivative vanishes indicate possible maxima or minima, and leave the necessary and sufficient conditions for maxima and minima and points of inflection for the calculus.

The transformation of coördinates is introduced quite incidentally; a very good plan. The student is impressed with the simplicity of the matter and learns it without undue stress.

The chapters dealing with the equation of a locus and the analytic treatment of geometric properties follow the usual plan. However, it is doubtful whether enough is said in Chapter 7 regarding empirical loci to make the introduction of the topic worth while.

The arrangement of the material of solid analytics follows closely that of plane analytics. Quadric surfaces are treated as loci of points, lines and conics which move subject to suitable conditions.

On the whole, the text presents the subject in a satisfactory manner and, if properly supplemented by informal lectures, should form a good basis for the work in analytics.

Analytic Geometry, Brief Course. By L. P. SICELOFF, G. A. WENTWORTH and D. E. SMITH. Boston, Ginn & Co. Price $1.80.

This text furnishes rather a contrast to the one discussed above. In fact, in the preface we read:

"This text has not been prepared for the purpose of exploiting any special theory of presentation; it aims solely to set forth the leading facts of the subject clearly, succinctly, and in the same practical manner that characterizes the other textbooks of the series. . . . Throughout the work a special effort has been made to present the material in so natural and simple a manner that the student can comprehend it largely through his own reading."

The locus of the equation is discussed briefly, 10 pages, and simply. Transformation of coördinates is given a short chapter as is also polar coördinates. The analytic treatment of the simple geometric properties of the conics is given at the close of the chapters on each of the conics. The general conic and the reduction of the general quadratic to standard form are discussed briefly in Chapter 10. The two chapters on solid analytics are short and to the point.

The plan of the text, mechanically, is typical of the other texts of the Wentworth-Smith series, with the free use of italics for theorems and important problems. The text is very readable and, in the opinion of the writer, should need little supplementing by lectures for the average class.

S. E. FIELD.

La Théorie de Bohr (Publications de la Société de Chimie-Physique, X). By E. BAUER. Paris, Hermann, 1922. 8vo. 52 pages. Price, paper cover, 4.50 francs.

This is an interesting summary of Bohr's theory of atomic structure as developed up to 1921. It includes brief explanations of the fundamental facts underlying the idea of the nuclear atom, its nuclear charge and dimensions, atomic number, etc. The relation of the electronic configurations to the physical and chemical properties is explained in some detail. Spectral series are developed in terms of the theory of quanta, with emphasis upon the Balmer series of hydrogen, the Stark effect and the X-ray spectra of the heavier elements.

E. F. BARKER.

NOTE ON A RECENT PUBLICATION.

We have already referred (*1921*, 79) to the publication of volume 14 of the magnificent edition of *Oeuvres Complètes* of Christian Huygens now being produced by the Dutch Society of Sciences. Professor D. J. Korteweg of the University of Amsterdam has for many years dedicated his life to the editing of the *Oeuvres*. Pages 1–163 of volume 15, edited by D. J. Korteweg and A. A. Nijland, have just been printed off. They contain "Recueil des observations astronomiques, 1657–1694." The usual avertissement by the editors occupies pages 3–53.

ARTICLES IN CURRENT PERIODICALS.

Acta Mathematica, volume 42, 1922: "Surface transformations and their dynamical applications" by G. D. Birkhoff, 1–119; "Mémoire sur les polynômes de Bernoulli" by N. E. Nörlund, 121–196; "Sur les fonctions hypersphériques et sur l'expression de la fonction hypergéométrique par une dérivée généralisée" by J. K. de Feriet, 197–207; "Über Konvergenz unendlicher Kettenbrüche mit durchweg reellen Elementen" by O. Szasz, 209–237; "A proof that every aggregate can be well-ordered" by P. E. B. Jourdain, 239–261; "Über die konforme Abbildung endlich und unendlich vielfach zusammenhängender symmetrischen Bereiche" by P. Koebe, 263–287; "Über die automorphen Functionen zweier Veränderlichen" by P. J. Myrberg, 289–318; "Développements asymptotiques des solutions d'une classe d'équations différentielles linéaires" by T. Carleman, 319–336.

Journal de Mathématiques Pures et Appliquées, ninth series, volume 1, part 4, 1922: "Nuove osservazioni su alcuni metodi d'approssimazione dell' analisi" by M. Pincone, 335–393; "Differential properties of functions of a complex variable which are invariant under linear transformations" by E. J. Wilczynski, 393–436.

L'École Normale Supérieure, Annales, volume 57, December, 1922: "Sur les formules d'interpolation de Stirling et de Newton" by N. E. Nörlund, 343–403.

Messenger of Mathematics, volume 52, July, 1922: "Electromagnetic potentials and radiation" by R. Hargreaves, 34–38; "Partial fractions associated with quadratic factors" by E. H. Neville, 39–42; "Polygons inscribed in one circle and circumscribed to another" by E. C. Titchmarsh, 42–45; "On curvature, tortuosity and higher flexures of a curve in flat space of n dimensions" by R. F. Muirhead, 45–48.

Quarterly Journal of Mathematics, volume 49, October, 1922: "Developable surfaces through given pairs of guiding curves" by E. H. Neville, 193–220; "On plane unicursal quintic curves with a triple point" by H. Hilton and Grace D. Sadd, 220–226; "Abstract definition of the symmetric and alternating groups and certain other permutation groups" by R. D. Carmichael, 226–283; "On the elementary treatment of the theory of least squares" by L. V. Meadowcroft, 283–288.

Rendiconti del Circolo Matematico di Palermo, volume 46, fasc. 2–3, 1922: "Questions d'analysis situs" by J. Chuard, 185–224; "Sul luogo dei piedi delle normali condotte da uno stesso punto alle curve d'un fascio" by A. Ajello, 225–231; "Sull'independenza di un integrale dal parametro" by G. Vivanti, 232–235; "A theorem on loci connected with cross-ratios" by J. L. Walsh, 236–248; "Sull'ubicazione dei punti di massima sollecitazione elastica tangenziale in un prisma retto, sollecitato a torsione" by E. Allara, 249–258; "Á propos de l'équation des télégraphistes" by E. Picard, 259–260; "Sopra un'equazione integrale" by P. Nalli, 261–264; "Sulla costruzione delle superficie iperellittiche cicliche" by M. Verzi, 265–307; "Sur une identité remarquable de la théorie des congruences binomes" by G. Rados, 308–314; "Il problema di Bianchi" by P. Tortorici, 315–345; "Zum Koebeschen Verzerrungssatz" by E. Landau, 347–348; "Zur additiven Primzahltheorie" by E. Landau, 349–356; "Su di un'equazione integrale di prima specie" by F. Tricomi, 357–387; "Analisi delle funzioni a variazione limitata" by G. Vitali, 388–408; "Sulle superficie con due famiglie di curve ortogonali deformabili in linee di livello e sopra una proprietà caratteristica delle superficie ad una minima" by C. Sansone, 409–436; "Sul calcolo delle variazioni" by M. Pincone, 437–455; "Über einen Bieberbachschen Satz" by E. Landau, 456–462; "L'equazione differenziale risolvente dell'equazione trinomia" by G. Belardinelli, 463–472.

TRANSACTIONS OF THE AMERICAN MATHEMATICAL SOCIETY, volume 23, no. 2, March, 1922: "Differential geometry of the complex plane" by J. L. Coolidge, 117–134; "Invariantive characterizations of linear algebras with the associative law not assumed" by C. C. Macduffee, 135–150; "Curves invariant under point-transformations of special type" by Mary F. Curtis [Mrs. W. C. Graustein], 151–172; "Die Zerlegung von Primzahlen in algebraischen Zahlkörpern" by A. Speiser, 173–178; "The elliptic modular functions associated with the elliptic norm curve E^7" by R. Woods, 179–197; "Linear equations with two parameters" by Anna J. Pell, 198–211; "The theory of functions of one Boolean variable" by K. Schmidt, 212–222.

ZEITSCHRIFT FÜR MATHEMATISCHEN UND NATURWISSENSCHAFTLICHEN UNTERRICHT, volume 53, nos. 9–10, published October 13, 1922: "Mathematische Lehrfilme" by M. Ebner, 193–200; "Die Brennpunktseigenschaften der Kegelschnitte, abgeleitet mit Hilfe der Kollineation" by E. Diehl, 200–210; "Freie und erzwungene harmonische Schwingungen (in elementarer Behandlung)" by H. Thorade, 211–218; "Zum 'Paradoxon der Gravitation'" by K. Bögel, 218–220; "Bemerkung zu vorstehender Abhandlung" by H. Teege, 220–221; "Beiträge zur Behandlung der Sätze vom Sehnenviereck und Tangentenviereck" by W. König, 222–223; "Noch eine Bemerkung zum Höhensatz von A. Maennersdorfer" by H. Hoyer, 223–224; "Die Newtonsche Formel für die Quadratur" by C. Herbst, 224–225; "Nochmal Dreikant und Polarkant" by C. Stengel, 225; "Aufgabenrepertorium," 226–232; "Bücherbesprechungen" and "Zeitschriftenschau," 243–248.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND NORMAN ANNING.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N. B. Problems containing results believed to be new, or extensions of old results, are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known text-books, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

3017. Proposed by F. W. PERKINS, JR., Cambridge, Mass.

M. Edouard Lucas, in *Récréations Mathématiques* (vol. 1, pp. 41–51), gives a proof of the following theorem: Any labyrinth can be traversed in such a way as to cover each path twice, and only twice, returning finally to the starting point (provided merely that one has some means of marking paths as they are traversed) by following the three rules given below:

(1) On arriving at a new vertex (*i.e.*, one not visited before), leave by a new path if possible; otherwise return by the same path.

(2) On arriving at an old vertex by a new path, return by the same path.

(3) On arriving at an old vertex by an old path, leave by a new path if possible; otherwise by a path marked just once. It can be shown that it is always possible to do this until every path of the labyrinth has been marked twice.

Assuming Lucas' results show that, if his rules are observed, the two trips on each path are in opposite directions.

NOTE: We may represent any labyrinth geometrically by a collection of points ("vertices") not necessarily in a plane which are joined by plane or twisted curves ("paths") in any manner whatever, provided merely that it is possible to pass from any point to any other along the curves and that the curves do not meet except at vertices.

3018. Proposed by J. G. COFFIN, New York City.

A platform 1000 feet long can move in either direction at any speed up to 2000 feet per second. A man is stationed on one end of the platform and is provided with a horn whose note-frequency is 100 vibrations per second. Assuming that the speed of sound is 1000 feet per second,

explain what occurs to an observer stationed on the other end of the platform when the horn is sounded under the following conditions: (*a*) Platform at rest; (*b*) moving to the right with a speed of 500 ft./sec.; (*c*) 1000 ft./sec.; (*d*) 1500 ft./sec.; (*e*) with the same three speeds but to the left.

What happens under similar conditions when the man with the horn is on the ground and the listener is on the platform and *vice-versa*?

3019. Proposed by J. B. REYNOLDS, Lehigh University.

Find the equation of the curve on the cylinder $x^2 - y^2 = a^2$ such that the tangent to it cuts the xy-plane in a lemniscate as the point of tangency moves along the curve.

3020. Proposed by JOHN NICHOLS, Portland, Oregon.

Rationalize $\alpha^{1/2} + \beta^{1/2} + \cdots + \mu^{1/2} + \nu^{1/2} + p = 0$.

3021. Proposed by N. ALTSHILLER-COURT, University of Oklahoma.

Prove that the two lines joining the points of intersection of two orthogonal circles to any point of one of them meet the other circle in two diametrically opposite points, and conversely.

3022. Proposed by M. J. SPINKS, Wilmington, Ohio.

Given that ABC is an equilateral spherical triangle right-angled at C, prove that $\sec A = 1 + \sec a$.

3023. Proposed by E. T. BELL, University of Washington, Seattle, Wash.

The equation $x^p + y^p + z^p = 0$ is possible in integers x, y, z prime to the odd prime p, if

$$\frac{1}{2}\left[\frac{1}{2}N_2(p) - \frac{1}{3}N_3(p) + \frac{1}{4}N_4(p) - \cdots + \frac{1}{p-1}N_{p-1}(p)\right] + 1$$

is divisible by p, where $N_r(n)$ is the number of representations (order essential) of n as a sum of r square integers with roots not equal to zero.

3024. Proposed by H. F. MACNEISH, College of the City of New York.

The angles of a triangle ABC are divided into n equal parts ($n = 3, 4, 5, \cdots$) and the two n-sectors of angles B and C which are adjacent to side BC intersect in A_1, the next two n-sectors in A_2 and so on to A_{n-1}. Points $B_1, B_2, B_3, \cdots C_1, C_2, C_3, \cdots$ are similarly determined. Which of the triangles $A_iB_iC_i$ ($i = 1, 2, 3, \cdots n - 1$) are equilateral?

SOLUTIONS

2907 [1921, 277]. Proposed by G. E. RAYNOR, Princeton University.

Sum the following series:

$$(a)\quad \sum_0^\infty \frac{1}{(2n+1)(3n+1)}; \qquad (b)\quad \sum_0^\infty \frac{1}{(3n+1)(3n+2)}.$$

SOLUTION BY OTTO DUNKEL, Washington University.

These two series are special cases of

$$\sum_{i=0}^{i=\infty} \frac{1}{(a_1 i + b_1)(a_2 i + b_2)} = \frac{1}{a_1 b_2 - a_2 b_1} \sum_{i=0}^{i=\infty} \left(\frac{1}{i + p_1} - \frac{1}{i + p_2}\right), \tag{1}$$

where the a's and b's are positive integers, *i.e.*, > 0; $a_1 b_2 - a_2 b_1 \neq 0$; and $p_1 = b_1/a_1$, $p_2 = b_2/a_2$.

Set

$$\sum_{i=0}^{i=m} \frac{1}{i+p} - \log\left(\frac{m+1+p}{p}\right) = C(m, p); \tag{2}$$

then, since $C(m, p)$ is positive and less than $\sum_{i=0}^{i=\infty} 1/2(i+p)^2$, it converges to a limit $\gamma(p)$ and we have

$$\sum_{i=0}^{i=m_1} \frac{1}{i+p_1} - \sum_{i=0}^{i=m_2} \frac{1}{i+p_2} = \log\left[\frac{1 + \dfrac{1+p_1}{m_1}}{1 + \dfrac{1+p_2}{m_2}} \cdot \frac{a_1 m_1}{a_2 m_2} \cdot \frac{b_2}{b_1}\right] + C(m_1, p_1) - C(m_2, p_2). \tag{3}$$

If we take $m_1 = m_2$, then we have as the value of (1)

$$\frac{1}{a_1b_2 - a_2b_1}\left[\log\frac{a_1b_2}{a_2b_1} + k\right], \tag{4}$$

where $k = \gamma(p_1) - \gamma(p_2)$.

The value of k may be found as follows: we have

$$\Sigma\left(\frac{1}{i + p_1} - \frac{1}{i + p_2}\right) = a_1\Sigma\frac{1}{a_1i + b_1} - a_2\Sigma\frac{1}{a_2i + b_2}.$$

Set

$$f(x) = a_1\sum_{i=0}^{i=\infty}\frac{x^{a_1i+b_1}}{a_1i + b_1} - a_2\sum_{i=0}^{i=\infty}\frac{x^{a_2i+b_2}}{a_2i + b_2}, \qquad 0 \leqq x < 1,$$

then

$$f'(x) = a_1\sum_{i=0}^{i=\infty}x^{a_1i+b_1-1} - a_2\sum_{i=0}^{i=\infty}x^{a_2i+b_2-1} = \frac{a_1x^{b_1-1}}{1 - x^{a_1}} - \frac{a_2x^{b_2-1}}{1 - x^{a_2}},$$

and hence

$$f(x) = a_1\int_0^x\frac{x^{b_1-1}}{1 - x^{a_1}}\,dx - a_2\int_0^x\frac{x^{b_2-1}}{1 - x^{a_2}}\,dx,$$

where the constant of integration is zero, since $f(0) = 0$.

Now $f(x)$ may be written as a *single* power series for $x < 1$ instead of the sum of two power series as above, the terms of the second series being combined with corresponding terms of the first or inserted in order between pairs of terms of the first. Thus, if $a_2m_2 + b_2$ is the exponent in a term of the second series, we can find a positive integer m_1 so that

$$a_1m_1 + b_1 \leqq a_2m_2 + b_2 < a_1(m_1 + 1) + b_1$$

if we assume, as we may, that $b_1 \leqq b_2$. Hence we see that

$$\operatorname{limit}\frac{a_2m_2}{a_1m_1} = 1,$$

as m_1 and m_2 become infinite. Now (3) shows that this power series for $x = 1$ converges and we have

$$\operatorname{limit}\left[\sum_{i=0}^{i=m_1}\frac{1}{i + p_1} - \sum_{i=0}^{i=m_2}\frac{1}{i + p_2}\right] = \log\left(\frac{b_2}{b_1}\right) + k.$$

But for $x < 1$ the power series has the value given above in integral form, and hence by a well-known theorem (see Goursat-Hedrick, *Mathematical Analysis*, vol. 1, p. 378)

$$\operatorname*{limit}_{x=1}\left[a_1\int_0^x\frac{x^{b_1-1}}{1 - x^{a_1}}\,dx - a_2\int_0^x\frac{x^{b_2-1}}{1 - x^{a_2}}\,dx\right] = f(1) = \log\left(\frac{b_2}{b_1}\right) + k. \tag{5}$$

It now follows from (1) and (4) and this result that

$$\sum_{i=0}^{i=\infty}\frac{1}{(a_1i + b_1)(a_2i + b_2)} = \frac{1}{a_1b_2 - a_2b_1}\left[f(1) + \log\left(\frac{a_1}{a_2}\right)\right],$$

where $f(1)$ is the limit given in (5). This shows that the parts of the integrals in (5) which become infinite for $x = 1$ cancel. This may also be shown directly by an independent method. This same method of summation may be applied whatever the number of factors, $ai + b$, in the denominator of (1) provided that all the determinants such as $a_1b_2 - a_2b_1$ are different from zero.

Applying this result to the given special cases, we have

$$(a)\quad \frac{\sqrt{3}}{6}\pi + \log\frac{3\sqrt{3}}{4}, \qquad (b)\quad \frac{\pi}{3\sqrt{3}}.$$

2924 [1921, 393]. Proposed by FLORENCE P. LEWIS, Goucher College.

Given a triangle and a conic. Through each vertex of the triangle there pass two lines harmonic to the tangents through that point and to the sides of the triangle. Prove that the six lines so found pass by threes through four points.

SOLUTION BY OTTO DUNKEL, Washington University.

The equations of the three sides of the triangle and of two straight lines through the vertices A and B, distinct from the sides, may be written in abridged notation as $\alpha = 0$, $\beta = 0$, $\gamma = 0$,

$\beta + \gamma = 0$, $\gamma + \alpha = 0$, by a suitable choice of α, β and γ. Hence we may write the equations of the six lines, separating the sides of the triangle harmonically, as

$$\begin{aligned} \beta + \gamma &= 0, \quad & \gamma + \alpha &= 0, \quad & \alpha + h\beta &= 0, \\ \beta - \gamma &= 0, \quad & \gamma - \alpha &= 0, \quad & \alpha - h\beta &= 0. \end{aligned}$$

Then each of the three pairs of the following lines:

$$\begin{aligned} \beta + p\gamma &= 0, \quad & \gamma + q\alpha &= 0, \quad & \alpha + rh\beta &= 0, \\ p\beta + \gamma &= 0, \quad & q\gamma + \alpha &= 0, \quad & r\alpha + h\beta &= 0, \end{aligned}$$

are also separated harmonically by the corresponding pair of the first set, for $2(\beta + p\gamma) = (1 + p)(\beta + \gamma) + (1 - p)(\beta - \gamma)$ and $2(p\beta + \gamma) = (1 + p)(\beta + \gamma) - (1 - p)(\beta - \gamma)$.

If the last set of six lines are tangent to a conic, then the hexagon obtained by taking them in the order written, *i.e.*, $\beta + p\gamma$, $p\beta + \gamma$, $\gamma + q\alpha$, etc., will have the diagonal lines

$$q\gamma - rh\beta = 0, \quad p\beta - q\alpha = 0, \quad r\alpha - hp\gamma = 0,$$

and these must meet in a point by Brianchon's theorem. This gives readily $h^2 = 1$ and on inserting either value of h in the first set, it will be seen at once that the six lines meet as stated in the problem.

Also solved by WILLIAM HOOVER, R. M. MATHEWS, J. R. MUSSELMAN, and H. L. OLSON.

2941 [1922, 28]. Proposed by W. D. CAIRNS, Oberlin College.

$1^2 + 2^2 + 3^2 + \cdots + (n - 1)^2$ is a function of n. Find its derivative with respect to n.

SOLUTION BY S. A. COREY, Des Moines, Iowa.

The given function of n, when developed by Euler's summation formula, reduces to the form $(2n^3 - 3n^2 + n)/6$, the derivative of which is $n^2 - n + 1/6$.

NOTE BY THE EDITORS—The sum $\sum_{i=1}^{i=n-1} i^2$ is a function of n which is defined only for integral values of n. It has then no derivative. In order to speak of a derivative, the function must be defined for other values of n. This may be done arbitrarily by finding the sum as a polynomial of the third degree in n and then defining the values of the function for all values of x as the values of this polynomial, $(2x^3 - 3x^2 + x)/6$; this the above solution does. But this is only one of many ways of defining a function $f(x)$ such that when $x = n$, a positive integer, $f(x) = \sum_{i=1}^{i=n-1} i^2$; for example, we might write

$$f(x) = A \sin^p \pi x + (2x^3 - 3x^2 + x)/6, \qquad p > 0,$$

and hence the derivative would vary with the definition of the function for non-integral values.

Also solved by T. M. BLAKSLEE, MICHAEL GOLDBERG, C. E. NORWOOD, E. J. OGLESBY, W. J. THOME, J. B. REYNOLDS, and F. L. WILMER.

2942 [1922, 28]. Proposed by L. E. DICKSON, University of Chicago.

I am dealt 13 cards at whist. What is the chance that all my cards will be diamonds?

SOLUTION BY E. W. WOOLARD, U. S. Weather Bureau, Washington, D. C.

The number of ways in which 52 cards may be dealt to four players giving a single card to the first, second, third, fourth and then repeating in this order until the cards are exhausted is 52! The number of ways in which a specified player may receive 13 diamonds and the other three no diamonds is 13! 39!. Hence the probability of the player receiving 13 diamonds is 13! 39!/52! or 1 in 635,013,559,600. Here the problem is treated as a problem in permutations, but it may also be considered by combinations giving the same result.

Also solved by A. BOGARD, S. A. COREY, E. J. OGLESBY, and J. B. REYNOLDS.

2943 [1922, 29]. Proposed by L. E. DICKSON, University of Chicago.

In the game of bridge, what is my chance: (*a*) that my hand will contain 4 aces? (*b*) that some hand will contain 4 aces? (*c*) that my hand will be a Yarborough, *i.e.*, contain no honor?

SOLUTION BY E. J. OGLESBY, Flushing, N. Y.

There are 52! ways in which 52 cards may be dealt one by one to four players in order. A specified player may receive four aces in 4! $({}_{13}C_4)$ or 13!/9! ways, and there are 48! ways in which the other three players may receive the remaining 48 cards. The probability for (*a*) is then

$$\frac{13!\,48!}{9!\,52!} = \frac{11}{4165},$$

while for (*b*) it is 44/4165.

(*c*) Webster's *New International Dictionary*, 1910, defines "Yarborough" as a hand containing no card higher than a nine. This requires that my hand shall be selected from the 32 cards which are nine or lower. There results the probability:

$$({}_{32}C_{13})/({}_{52}C_{13}) = (32!\,39!)/(52!\,19!) = 0.00054.$$

Also solved by S. A. COREY, J. B. REYNOLDS, and E. W. WOOLARD.

NOTE BY THE EDITORS—Part (*c*) was interpreted differently by some of the solvers with a corresponding difference in the numerical result.

2944 [1922, 29]. Proposed by S. A. COREY, Des Moines, Iowa.

A particle of mass m, starting from rest, is drawn by a string over a smooth horizontal plane, the other end of the string moving in the plane with uniform acceleration n along a line perpendicular to the initial position of the string. Prove that the tension of the string is $3mn\cos\theta$, where θ is the angle which the string makes with the given line. Also prove that the motion of the particle is vibratory.

SOLUTION BY J. B. REYNOLDS, Lehigh University.

The acceleration of the particle may be considered as made up of three components, n parallel to the given line, $a\ddot{\theta}$ perpendicular to the string and $a\dot{\theta}^2$ along the string. Resolving perpendicular to the string, we have $n\sin\theta + a\ddot{\theta} = 0$, and upon integrating,

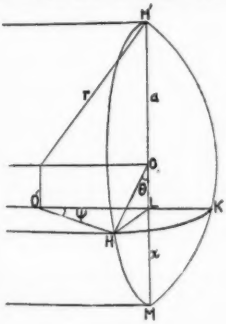
$$-n\cos\theta + \tfrac{1}{2}a\dot{\theta}^2 = c = 0.$$

Therefore, $\dot{\theta} = \mp\sqrt{\dfrac{2n\cos\theta}{a}}$ and the string oscillates between $\theta = \pm\dfrac{\pi}{2}$. Again, resolving along the string, we have, T being the tension, $T = m(a\dot{\theta}^2 + n\cos\theta) = m(2n\cos\theta + n\cos\theta) = 3mn\cos\theta$.

Also solved by F. L. WILMER.

2947 [1922, 29]. Proposed by D. H. MENZEL, Princeton University.

An oil tank has the shape of a cylinder with ends which are segments of a sphere and with horizontal axis. The diameter of the cylinder being given, and the radius of the spherical segments, derive a formula that will express the volume of the liquid contained in the tank in terms of its depth.



SOLUTION BY J. B. REYNOLDS, Lehigh University.

Let l be the length of the cylindrical portion of the tank, a its radius, and r the radius of the spherical caps. Let the distance of a horizontal section from the bottom of the cylinder be $x = ML = a(1-\cos\theta) = 2a\sin^2\theta/2$, where MOM' is the vertical diameter of a cylinder end. The horizontal section intersects this circular end in a chord HLH' which subtends the angle 2θ at the center O, such that $\theta = \angle\, LOH$, and the spherical cap in a segment $HLH'K$ with the radius $O'K$. Let $\psi = \angle\, KO'H$, then $O'L = \sqrt{r^2 - a^2} = a\lambda$, $O'K = \sqrt{r^2 - a^2\cos^2\theta} = a\lambda\sqrt{1+\sin^2\theta/\lambda^2}$, and $\tan\psi = \sin\theta/\lambda$.

The section consists of a rectangle of area $2al\sin\theta$ and two segments of a circle each of area $a^2(\lambda^2 + \sin^2\theta)\psi - a^2\lambda\sin\theta$. Set-

ting $dx = a \sin\theta d\theta$, the volume is given by the integral

$$V = 2a^3 \int_0^{\pi} \left[(l - a\lambda) \sin^2\theta + a(\lambda^2 \sin\theta + \sin^3\theta) \tan^{-1}\left(\frac{\sin\theta}{\lambda}\right) \right] d\theta$$

$$= a^2\left(l - 2\lambda a - \frac{4}{3}\lambda^3 a\right)\theta + a^2\left(\frac{2}{3}a\lambda - \frac{l}{2}\right)\sin 2\theta$$

$$+ \frac{a^3}{3}\cos\theta[\cos 2\theta - 6\lambda^2 - 5] \tan^{-1}\left(\frac{\sin\theta}{\lambda}\right) + \frac{4a^3}{3}(\lambda^2+1)^{3/2} \tan^{-1}(\sqrt{\lambda^2+1}\tan\theta/\lambda)$$

where $\theta = 2\sin^{-1}\sqrt{x/2a}$, $0 \leqq \theta \leqq \pi$, and $0 \leqq \tan^{-1}\left(\frac{\sqrt{\lambda^2+1}}{\lambda}\tan\theta\right) \leqq \pi$. When $\theta = 0$ or π, $\tan^{-1}(\sin\theta/\lambda) = 0$. When $\lambda = 0$, $\tan^{-1}(\sin\theta/\lambda) = \pi/2$, $\tan^{-1}(\sqrt{\lambda^2+1}\tan\theta/\lambda) = 0$.

Also solved by MOE BUCHMAN, P. FITCH, MICHAEL GOLDBERG, and H. W. REDDICK.

2948 [1922, 29]. Proposed by J. B. REYNOLDS, Lehigh University.

Find the envelope of the normal planes to the curve,

$$x = a\cos t, \qquad y = a(1 - \cos t), \qquad \text{and} \qquad z = a\sin t.$$

SOLUTION BY MAURICE BAUDIN, Miami University.

Translating the coördinate axes to $(0, a, 0)$ as origin and rotating them about the z-axis through $-\pi/4$, we write the equations of the given curve (a plane curve)

$$x = \sqrt{2}a\cos t, \qquad y = 0, \qquad z = a\sin t.$$

The equation of the normal is

$$Z - \sqrt{2}\tan t\, X + a\sin t = 0.$$

Eliminating t from this equation and $-\sqrt{2}\sec^2 t\, X + a\cos t = 0$, we obtain the equation of a cylinder

$$(\sqrt{2}\,X)^{2/3} + Z^{2/3} = a^{2/3}.$$

2950 [1922, 29]. Proposed by T. M. SIMPSON, JR., Randolph-Macon College, Ashland, Va.

Determine the curve which cuts the radius vector at an angle proportional to the radius vector.

SOLUTION BY A. H. WILSON, Haverford College.

The differential equation of the curve is

$$\frac{rd\theta}{dr} = \tan kr,$$

where k is the factor of proportionality, and the angle in question is measured in the positive direction from the radius vector to the tangent.

On expanding $\tan kr$, an integration in series may be effected:

$$\theta = c + kr + k^3r^3/9 + \cdots;$$

which is convergent when $r < \pi/2k$.

There are infinitely many limiting or singular curves of the solutions, namely, the concentric circles $r = (2n+1)\pi/2k$.

These circles are included among the solutions.

Inside such a circle and near it $dr/d\theta > 0$; while outside $dr/d\theta < 0$. Hence r is increasing with θ just inside the circle and decreasing as θ increases just outside; that is, a curve in any one of the rings defined by the concentric circles wraps around (approaches) both the inner and the outer circle for increasing θ.

It is clear from the series solution that any curve in the first circle cuts a radius infinitely many times; and the like is true for the curves of any ring. The curves therefore approach the circles asymptotically.

On the circles $r = n\pi/k$ the curves are radial in direction, and the sense of variation of r changes at these points.

The origin is a "ray-point."

These results are for the most part also clear from geometric considerations.

Also solved by WILLIAM HOOVER who studied the pedal curve.

2951 [1922, 81]. Proposed by B. F. FINKEL, Drury College.

What is the average linear velocity of a point on the periphery of a locomotive driving wheel 2 feet in diameter and making 100 revolutions per minute?

SOLUTION BY H. S. UHLER, Yale University.

The following solution is offered because it introduces generality but does not involve any more kinematical principles than the given numerical problem.

Let a, v, and ω denote respectively the radius, the rectilinear velocity, and the angular velocity of the wheel. Let 2θ denote the angle which the radius to the point on the periphery makes, at any instant, with the upper half of the vertical diameter. Then the angle which the tangential velocity $a\omega$ makes with the rectilinear velocity v equals 2θ. By any rule for compounding vectors it will be found that the magnitude of the resultant velocity of the moving point is represented by

$$(v^2 + a^2\omega^2 + 2va\omega \cos 2\theta)^{1/2}.$$

To find the mean value V_m of this expression it is sufficient to integrate 2θ only over a vertical semicircle, since the magnitude of the resultant velocity is the same for the two points of any pair on the same horizontal line or level. Hence

$$V_m = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} (v^2 + a^2\omega^2 + 2va\omega \cos 2\theta)^{1/2} d\theta$$

$$= \frac{2}{\pi}(v + a\omega)\int_0^{\frac{1}{2}\pi} d\theta \sqrt{1 - k^2 \sin^2\theta},$$

where $k^2 \equiv 4va\omega/(v + a\omega)^2$.

When v does not equal $a\omega$ (slipping, skidding, etc.), k^2 is less than unity and V_m equals

$$(v + a\omega)\left[1 - \left(\frac{1}{2}\right)^2 k^2 - \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \frac{k^4}{3} - \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 \frac{k^6}{5} - \cdots\right].$$

See B. O. Peirce's "*A Short Table of Integrals*," No. 525, elliptic integral E.

When $v = a\omega$ the motion is pure rolling, $k^2 = 1$, and $V_m = 4a\omega/\pi$.

In the numerical problem $a = 1$ ft., $\omega = 10\pi/3$ radians per sec., hence $V_m = 40/3$ ft. per sec. $= 100/11$ mi. per hr.

Also solved by PHILIP FITCH, MICHAEL GOLDBERG, WILLIAM HOOVER, A. PELLETIER, W. J. THOME, J. B. REYNOLDS, F. L. WILMER and the PROPOSER.

NOTE BY THE EDITORS.—If the problem is interpreted "What is the average of the velocities represented as vectors?" it is evident that the average is v since diametrically opposite velocities $a\omega$ cancel.

Professor Hoover and F. L. Wilmer assumed the number of points proportional to the length of the path of the point considered and got an answer of $\frac{1}{2}(5\pi^2)$ feet per second.

An interesting point about this problem is the fact that a very similar one is proposed in a text-book of physics by a very prominent physicist. The book does not apparently assume any knowledge of calculus on the part of the student. What method did the author have in mind?

2955 [1922, 81]. Proposed by the late L. G. WELD.

Find the proportions of an anchor ring such that its section by a plane parallel to its axis and tangent to its inner circle (circle of the gorge) shall be a lemniscate.

SOLUTION BY H. C. BRADLEY, Massachusetts Institute of Technology.

Let r be the radius of the generating circle of the anchor ring (called also a *tore* or *torus*), and R the radius of the inner circle. The tangent plane to the surface at a point 0 of this latter circle

cuts it in a curve having the form of the figure 8. The indicatrix of the surface at this point must be an hyperbola and the two principal sections are those of the meridian plane at 0 and the plane of the inner circle. Hence the principal radii of curvature are r and R. Also the two tangents to the curve at the double point 0 are the tangents to the asymptotic lines at 0 and it follows from the equation of the indicatrix that the slopes of the two tangents are $\pm\sqrt{r/R}$.[1] In order to be a lemniscate the two tangents must be at right angles, and hence $r = R$.

[1] See *Mathematical Analysis*, Goursat-Hedrick, vol. 1, pages 501–507.

To show that this condition is both necessary and sufficient, we now derive the equation of the curve. Let the x-axis be the tangent to the inner circle, with the origin at 0, and take any point $P(x, y)$ on the curve. The distance of P from the axis of the anchor ring is $\sqrt{x^2 + R^2}$, and P lies on a circle of radius r in the meridian plane through P. Hence from the equation of this circle follows $[\sqrt{x^2 + R^2} - (R + r)]^2 + y^2 = r^2$ or

$$(x^2 + y^2)^2 = 4(R + r)(rx^2 - Ry^2).$$

If $r = R = a/2$, the equation becomes

$$(x^2 + y^2)^2 = 2a^2(x^2 - y^2)$$

which is that of a lemniscate.

Also solved by A. BOGARD, MICHAEL GOLDBERG, WILLIAM HOOVER, A. PELLETIER and F. L. WILMER.

NOTES AND NEWS.

It is hoped that readers of the MONTHLY will coöperate in contributing to the general interest of this department by sending items to R. W. BURGESS, Brown University, Providence, R. I.

Professor E. G. Bill, of Dartmouth College, has been appointed acting dean during the absence of Dean Laycock.

At New Hampshire State College, the following have been appointed instructors of mathematics: Mr. W. E. Wilbur, Mr. Hubert Huntley, and Mr. P. F. Howard. Professor H. L. Slobin, head of the department of mathematics, has been reappointed Director of the Summer School for the 1923 session.

Miss Eleanor P. Cushing has been made professor emeritus at Smith College.

Mr. R. S. Kimball has been appointed a member of the department of mathematics at the State Normal School at Worcester, Mass.

Mr. W. F. Shields has been appointed professor of freshman mathematics at Fordham University. He succeeds Rev. L. J. McGarry, who is now at Woodstock College, Maryland.

At Columbia University, Professor T. S. Fiske, of the department of mathematics, and Professor C. L. Poor, of the department of celestial mechanics, have been granted leave of absence for the present semester.

At Rutgers College, Professor A. A. Titsworth, formerly of the department of civil engineering, has been transferred to the department of mathematics as a full professor. Mr. W. B. Campbell, a graduate of the University of Pennsylvania, has been appointed instructor of mathematics. Professor Richard Morris read a paper on "The Cyclic Quadrilateral, a Recreation," at the October meeting of the Mathematical Association of New Jersey.

Miss Ruth Thompson has been appointed instructor of mathematics at the New Jersey College for Women.

Instructor J. T. Fairchild, of Ohio Northern University, has been promoted to a professorship of mathematics.

Associate Professor Mary E. Sinclair, of Oberlin College, is on leave of absence during the second semester of 1922–23, as holder of the Pratt fellowship of the National Association of University Women, and is studying at the University of Chicago.

Rev. Paul Muehlman, formerly head of the department of mathematics at Marquette University, has been transferred to Loyola University, Chicago, where he is holding a similar position.

Assistant professor B. F. Dostal, of the University of Denver, has been appointed assistant professor of mathematics at the Bradley Polytechnic Institute.

Mr. H. I. Mayes has been appointed a member of the department of mathematics at Blackburn College, Carlinville, Ill.

Mr. J. B. Hobbs has been appointed a member of the department of mathematics at the State Normal School, Oshkosh, Wisconsin.

At St. Mary's College, St. Mary, Kentucky, Mr. Charles McCarthy and Mr. Joseph Walsh have been appointed professors of mathematics.

Professor J. N. Mallory, of Union University, Jackson, Tenn., has returned to his work after a leave of absence of one and a half years, which he spent at Peabody College.

Mr. A. L. Hill has been appointed head of the department of mathematics at the State Normal School and Teachers College, Peru, Nebraska.

Mr. L. V. Robinson has been appointed head of the department of mathematics at Meridian College, Meridian, Texas.

Miss Paula Henry has succeeded her sister, Miss Phillis Henry, as director of the department of mathematics at Kidd-Key College, Sherman, Texas. Miss Phillis Henry has accepted a position as instructor at the University of Texas.

Miss Emma F. Barbe has been appointed instructor of mathematics at Westmoorland College, San Antonio, Texas.

At the Texas Agricultural and Mechanical College, Assistant Professors D. C. Jones and W. L. Porter have been promoted to associate professorships of mathematics, Mr. W. L. Hughes and Mr. P. K. Smith have been appointed assistant professors, and Mr. C. E. McCurry instructor.

At the University of Denver, Professor G. W. Gorrell of the Colorado State School of Mines has been appointed associate professor, and Dr. E. Frances Seiler of the University of Illinois instructor.

Mr. V. F. Morse, a graduate of the California Institute of Technology, has been appointed instructor of mathematics and mechanical drawing at Occidental College, Los Angeles, California.

Mr. P. M. Iloff has been head of the mathematics department of the State Teachers and Junior College at Chico, Cal., since January, 1922.

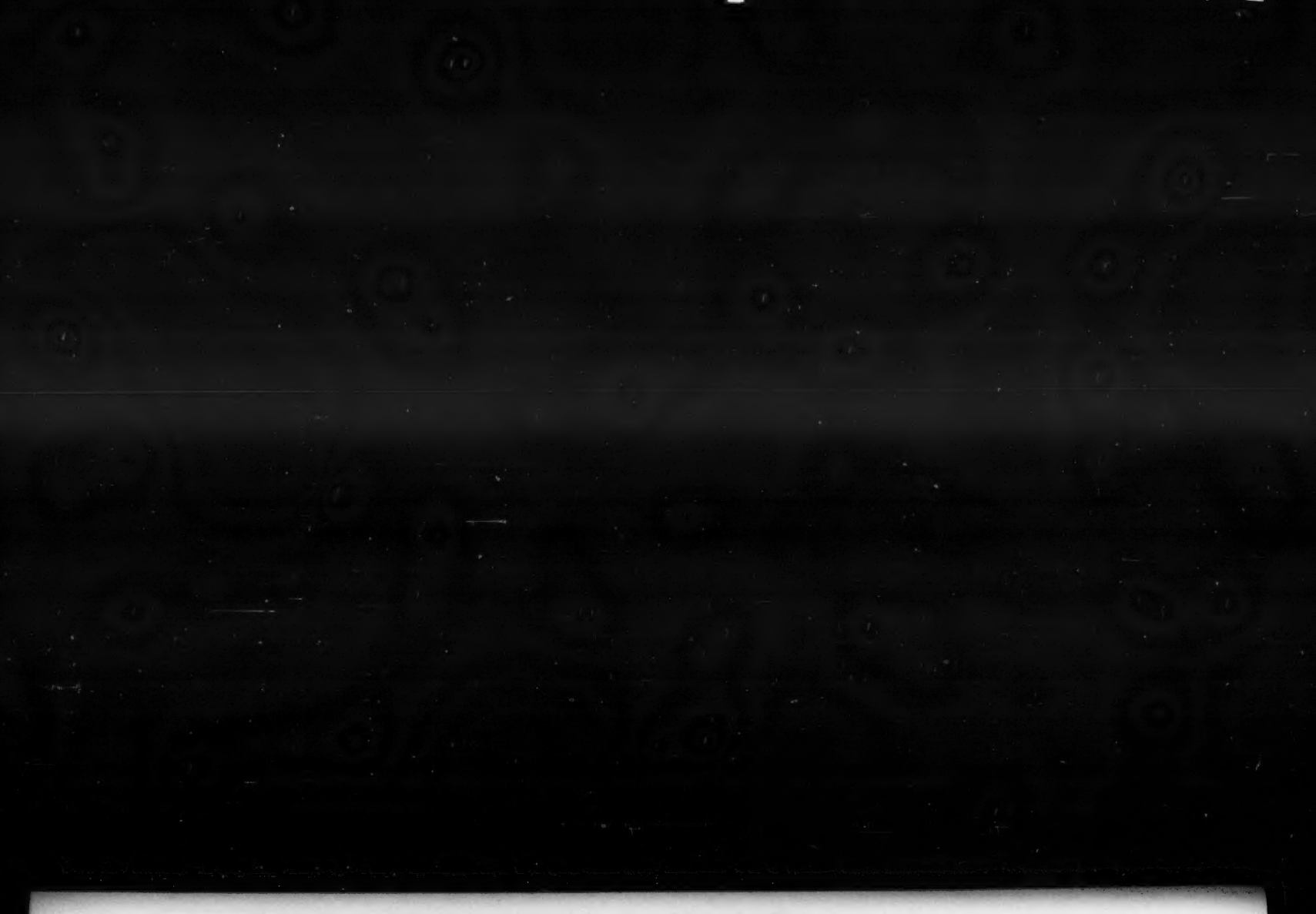
The Royal Astronomical Society has awarded its gold medal to Professor

A. A. Michelson, of the University of Chicago, for his application of the interferometer to astronomical measurements.

ANDREĬ ANDREEVICH MARKOV, professor of mathematics in the University of St. Petersburg from 1886 till 1907 when he was made professor emeritus, died July 27, 1922. He was born at Rjasan, Russia, June 2/14, 1856, and was a student at the University of St. Petersburg where he received the degree of doctor of mathematical sciences in 1884. He was a privatdocent in the University 1880–1886. His brilliant achievements won him election to the Academy of Sciences in St. Petersburg when he was only thirty years of age. He was one of the two members of this Academy which edited the *Oeuvres* (published in 1899 and 1907) of his former master, Tchebychef. His popular work on the calculus of probability first appeared at St. Petersburg in 1900, and a German translation, by H. Liebmann, of the second edition (1908), appeared at Leipzig in 1912. His calculus of finite differences first published at St. Petersburg in 1891 was also translated into German (Leipzig, 1896). Long lists of his published papers, beginning in 1879, may be found in "Poggendorff"; in the Royal Society *Catalogue of Scientific Papers*, volumes 10, 12, and 17; and in *Revue Semestrielle des Publications Mathématiques*. Brief biographical notes and a portrait are given in *Acta Mathematica, 1882–1912. Table Générale des Tomes 1–35*, Upsala, 1913.

ANTONIO FAVARO, distinguished professor of graphical statics and history of mathematics in the University of Padua since 1872, died September 30, 1922. He was born at Padua, May 21, 1847. He was the author of over 400 papers, memoirs, and books and was the acknowledged authority on the life and works of Galileo whose work he edited in the great national edition (20 volumes, Florence, 1890–1909). For this he was made "nobile." His most recent volume was the finely illustrated *L'Università di Padova* (Venice, 1922) distributed to delegates to the celebration of the seven hundredth anniversary of the founding of the University of Padua (*1922*, 318). His *Lezioni di statica Grafica* (Venice, 1877) was translated into French (2 volumes, Paris, 1879, 1885).

(*Published July 31, 1923.*)



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THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

EDITED BY

WALTER BURTON FORD, Editor-in-Chief
HERBERT ELLSWORTH SLAUGHT
JULIAN LOWELL COOLIDGE

WITH THE COÖPERATION OF

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THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XXX, 1923
NUMBER 5, JULY-AUGUST

PUBLISHED BY THE ASSOCIATION
LANCASTER, PA., AND ANN ARBOR, MICH.

Entered at the Post Office at Lancaster, Pa., as Second Class Matter
PUBLISHED MONTHLY

$4.00 a Year, Single Copies 45 cents, to Members;
$5.00 a Year, Single Copies 60 cents, to Others